

Vanishing of one dimensional L^2 -cohomologies of loop groups *

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Abstract

Let G be a simply connected compact Lie group. Let $L_e(G)$ be the based loop group with the base point e which is the identity element. Let ν_e be the pinned Brownian motion measure on $L_e(G)$ and let $\alpha \in L^2(\wedge^1 T^* L_e(G), \nu_e) \cap \mathbb{D}^{\infty, p}(\wedge^1 T^* L_e(G), \nu_e)$ ($1 < p < 2$) be a closed 1-form on $L_e(G)$. Using results in rough path analysis, we prove that there exists a measurable function f on $L_e(G)$ such that $df = \alpha$. Moreover we prove that $\dim \ker \square = 0$ for the Hodge-Kodaira type operator \square acting on 1-forms on $L_e(G)$.

1 Introduction

Let (M, g) be a compact Riemannian manifold. Let d be the exterior differential operator on M . Let d^* be the adjoint operator of d in the L^2 space of differential forms with respect to the Riemannian volume. Let $\square = dd^* + d^*d$. Celebrated Hodge-Kodaira theorem asserts that $\dim \ker \square|_p = b_p$. Here $\square|_p$ denotes the Hodge-Kodaira operator on the space of p -forms and b_p is the (real coefficient) Betti number of M . This theorem does not hold any more in non-compact Riemannian manifold. On the other hand, in infinite dimension, there exist natural measures, such as (pinned) Brownian motion measures, on spaces of paths over a Riemannian manifold. Several researchers have been trying to establish a differential geometry and analysis including Hodge-Kodaira type theorem based on Brownian motion measures. Since the path space $P_x(M) = C([0, 1] \rightarrow M \mid \gamma(0) = x)$ has trivial topology, one natural guess is that there are no harmonic forms on $P_x(M)$ except 0-dimension. When M is a Euclidean space and $x = 0$, the path space with the Brownian motion measure is the Wiener space. The notion of H -derivative fits in with the differential calculus based on the Wiener measure and Sobolev spaces are defined according to the H -derivative. However the vanishing of L^2 cohomologies in the Sobolev space category is not trivial because smooth functions in the sense of H -derivative need not to be smooth in the sense of Fréchet. The vanishing theorem on Wiener space was proved by Shigekawa [33] in the setting of Sobolev spaces.

When M is a general Riemannian manifold, the Bismut tangent space is used to define a vector field and H -derivative on $P_x(M)$. The Bismut tangent space appeared naturally in the study of integration by parts formula and the quasi-invariance of (pinned) Brownian motion measures [9]. This tangent space depends on the choice of the metric connection on M and if the curvature does not vanish, then the Lie bracket of the vector fields do not belong to the

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Bismut tangent space. This shows a difficulty to study exterior differential operators on $P_x(M)$. We refer the reader to [10, 27] for this problem. Let us consider a special case where M is a compact Lie group G . Since the curvature of the right (or left) invariant connection of G is 0, the Bismut tangent space of $P_e(G)$ which is defined by the right (or left) invariant connection is stable under the Lie bracket and the exterior differential operator on $P_e(G)$ is well-defined. Here e is the identity element. We note that Hodge-Kodaira's theorem on $P_e(G)$ was studied in [12] using Shigekawa's result on a Wiener space.

Now let us consider the pinned case. Let $L_x(M) = C([0, 1] \rightarrow M \mid \gamma(0) = \gamma(1) = x)$. We have difficulties for the definition of the exterior differential operator similarly to $P_x(M)$. Instead of working on $L_x(M)$, some researchers studied differential calculus over submanifolds in the Wiener space [5, 23, 1, 31]. Typical submanifolds are obtained by solutions of stochastic differential equations (=SDEs) on M . See (2.2). The tangent space of the submanifold is defined to be a closed subspace of the Cameron-Martin subspace of the Wiener space and the Lie brackets of vector fields on the submanifold are also vector fields on the submanifold. That is, the exterior differential operator is well-defined. In a certain case, since the submanifold is isomorphic in some sense to $L_x(M)$ which has non-trivial topology, one may expect that the dimension of harmonic forms on the submanifold coincides with the Betti number of $L_x(M)$. Note that solutions of SDE are smooth in the sense of H -derivative (or in the sense of Malliavin calculus) but generally discontinuous functional of Brownian motions. Hence these submanifolds are not submanifolds in usual sense and the link between the analysis over the submanifolds and the "topology" of them are very unclear subject. Nevertheless, Kusuoka succeeded in proving a Hodge-Kodaira theorem and announced positive results in [24]. See [25, 26] also. We explain his results in Section 2 briefly.

In the present paper, we study a Hodge-Kodaira theorem for 1-forms on the based loop group $L_e(G)$, where G is a compact Lie group. The exterior differential operator d on $L_e(G)$ is defined using the right (or left) invariant connection in the similar manner to $P_e(G)$. When G is simply connected, $\pi_2(G) = 0$ and so $\pi_1(L_e(G)) = 0$ and the first Betti number is 0. Therefore one may conjecture a vanishing theorem of "the Hodge-Kodaira operator" acting on 1-forms on $L_e(G)$. Indeed, this is one of the main results of this paper. Our proof of vanishing theorem is different from Kusuoka's ones. Here we explain the outline of our proof. First, we show that if α is a closed 1-form on $L_e(G)$, then there exists a function f on $L_e(G)$ such that $df = \alpha$. To show this, using a map from a Wiener space to $L_e(G)$, we change the problem to a problem on an "open subset" \mathcal{D}_ε of the Wiener space. The map is given by a solution of an SDE on G and a "retraction map" on the Wiener space. The "open subset" \mathcal{D}_ε is homotopy equivalent to $L_e(G)$ in some sense. The property of "open" should be understood in the sense of rough path analysis. The topology in the rough path analysis is finer than the usual uniform convergence topology of the Wiener space and the solution of SDE can be viewed as a continuous functional with respect to the topology. The most important next step is to establish a Poincaré's lemma on a ball-like set $U_r(\varphi)$ in the sense of rough path analysis. That is, we prove that a closed 1-form on $U_r(\varphi)$ is exact. Note that \mathcal{D}_ε has a countable cover by the ball-like sets. In the third step, using the topological property of $\pi_1(L_e(G)) = 0$, we prove that a closed 1-form on \mathcal{D}_ε is exact putting together the locally established Poincaré's lemma on $U_r(\varphi)$. Applying this, for any closed 1-form on $L_e(G)$, we can show the existence of f such that $df = \alpha$. Finally, using this result, hypoellipticity of Bochner Laplacian and essential self-adjointness of Hodge-Kodaira operator on $L_e(G)$, we can get our vanishing theorem.

The paper is organized as follows. In Section 2, we state main results in this paper and

make some remarks. In Section 3, we recall the necessary results in rough path analysis. We fix a subset Ω of d -dimensional Wiener space W^d on which Brownian rough path is defined. Then a version of the solution of SDE on a compact Lie group G can be defined for all $w \in \Omega$. Also we give necessary estimates for iterated integrals and Wiener integrals which will be used in Section 4. In Section 4, we introduce subsets $U_{r,\varphi}$, $U_r(\varphi)$ and prove a Poincaré's lemma for closed 1-forms on the subsets in Theorem 4.6 and Theorem 4.7. This kind of Poincaré lemma was studied by Kusuoka [26]. Also Shigekawa [36] studied Hodge-Kodaira operator with absolute boundary condition on convex domains in Wiener spaces. We note that $U_r(\varphi)$ is not an H -convex domain and the Poincaré lemma is non-trivial. To prove Theorem 4.6 and Theorem 4.7, we prove Poincaré's inequalities on finite dimensional approximation of $U_{r,\varphi}$ in Claim 2 in the proof of Theorem 4.6. The point is that the Poincaré constant is independent of the dimensions. At the end of this section, we introduce subsets S , \mathcal{D}_ε of Ω . S is a “submanifold” of Ω and isomorphic to $L_e(G)$ by the solution of the SDE on G . Note that Ω is not a linear space and S is not a submanifold in usual sense. The subset \mathcal{D}_ε is a kind of “tubular neighborhood” of S in Ω . In Section 5, we prove that \mathcal{D}_ε is covered by a countable family of $U_r(\varphi)$. In Section 6, we introduce notions of H -connectedness and H -simply connectedness. We prove that \mathcal{D}_ε is an H -connected and H -simply connected set when G is simply connected. This and Stokes theorem (Lemma 6.6) are used to prove the existence of a function F such that $dF = \beta$ for a closed 1-form β on \mathcal{D}_ε . In Section 7, we prove several results which are necessary for reducing the problem on $L_e(G)$ to that on \mathcal{D}_ε . First, we state relations between Sobolev spaces on S and $L_e(G)$. Next, we define a retraction map from \mathcal{D}_ε onto S . This kind of retraction map are used in [6, 18, 1]. We obtain a closed form on \mathcal{D}_ε by the pull-back of a closed form on $L_e(G)$ using the retraction map. We apply results in Section 4 to this closed form. In Section 8, we prove our main theorems.

2 Statement of results and remarks

Let W^d be the set of continuous paths on \mathbb{R}^d defined on $[0, 1]$ starting at 0. We denote by μ the Wiener measure on W^d whose Cameron-Martin subspace is $H = H^1([0, 1] \rightarrow \mathbb{R}^d \mid h_0 = 0)$. We recall the definition of Sobolev spaces ([22]) over the Wiener space (W^d, H, μ) . Let $\mathfrak{F}C_b^\infty(W^d, E)$ be the set of all smooth cylindrical functions with values in a separable Hilbert space E . When $E = \mathbb{R}$, we may omit E . We denote by $\mathbb{D}^{k,p}(W^d, E)$ the set of L^p functions with respect to μ on W^d with values in E which are k -times H -differentiable and all their derivatives are also in $L^p(\mu)$. We write $\mathbb{D}^\infty(W^d, E) = \cap_{k \geq 0, p > 1} \mathbb{D}^{k,p}(W^d, E)$. Let G be a compact Lie group and consider a bi-invariant Riemannian metric on G . Let $P_e(G)$ be the set of continuous paths which are defined on the time interval $[0, 1]$ and the starting point is e . Let $L_e(G)$ be the subset of $P_e(G)$ which consists of paths whose end points are also e . Let ν, ν_e be the Brownian motion measure on $P_e(G)$ and the pinned Brownian motion measure on $L_e(G)$ respectively. These measures are defined by the diffusion semigroup $e^{t\Delta/2}$, where Δ is the Laplace-Beltrami operator which is defined by the bi-invariant Riemannian metric. Let $T_e(G) = \mathfrak{g}$ be the Lie algebra of G . We identify it as the set of right invariant vector fields. The bi-invariant Riemannian metric defines an inner product on \mathfrak{g} . We fix an orthonormal basis $\{\varepsilon_1, \dots, \varepsilon_d\}$ which enables us to identify \mathfrak{g} and \mathbb{R}^d , where $d = \dim G$. Therefore we identify H and a set of H^1 -paths over \mathfrak{g} starting at 0 in this way. Set $H_0 = \{h \in H \mid h_1 = 0\}$. We recall the definition of H -derivative on $P_e(G)$ and $L_e(G)$. For a smooth cylindrical function $F(\gamma)$ on $P_e(G)$ (or $L_e(G)$), we define the

H -derivative of F to be a measurable map $G = G(\gamma)$ (actually smooth map in this case) from $P_e(G)$ (or $L_e(G)$) to H^* (or H_0^*) which satisfies

$$(G(\gamma), h) = \lim_{\varepsilon \rightarrow 0} \frac{F(e^{\varepsilon h} \gamma) - F(\gamma)}{\varepsilon}$$

for all $h \in H$ (or $h \in H_0$), where (\cdot, \cdot) is the pairing of the elements of H^* (or H_0^*) and H (or H_0). We denote $G(\gamma)$ by $dF(\gamma)$. This derivative corresponds to the derivative which is defined by a right-invariant vector field X_h on $L_e(G)$. The tangent space $T_\gamma L_e(G)$ is defined to be the set of all continuous mappings h from $[0, 1]$ to TG with $h(t) \in T_{\gamma(t)}G$ and $(R_{\gamma(\cdot)})_*^{-1}h(\cdot) \in H_0$. Here $R_a b = ba$ for $a, b \in G$. Naturally, $T_\gamma L_e(G)$ can be identified with H_0 . Therefore $\otimes^p T_\gamma^* L_e(G), \wedge^p T_\gamma^* L_e(G)$ can be identified with $\otimes^p H_0^*, \wedge^p H_0^*$ respectively. Accordingly, measurable covariant tensor fields, differential forms on $L_e(G)$ are defined to be measurable maps from $L_e(G)$ to $\otimes^p H_0^*, \wedge^p H_0^*$ respectively. The set $L_e(G)$ is a Banach manifold and there is a natural definition of the (co)tangent bundle. In this paper, we do not use the structure but use the derivative in the H -direction and the notation $T^* L_e(G)$ should be understood in such a sense.

To define Sobolev spaces of tensors over $L_e(G)$, we use the Levi-Civita covariant derivative ∇ which is defined using the right invariant Riemannian metric. The covariant derivative ∇ is a mapping on the smooth cylindrical tensor fields such that $\nabla T \in \mathfrak{F}C_b^\infty(\otimes^{p+1} T^* L_e(G))$ for $T \in \mathfrak{F}C_b^\infty(\otimes^p T^* L_e(G))$ ($p = 0, 1, 2, \dots$). The Sobolev space $\mathbb{D}^{k,q}(\otimes^p T^* L_e(G), \nu_e)$ ($k \in \mathbb{N} \cup \{0\}, q \geq 1$) is the completion of $\mathfrak{F}C_b^\infty(\otimes^p T^* L_e(G))$ by the norm $\| \cdot \|_{k,q}$ such that

$$\|T\|_{k,q} = \left(\sum_{i=0}^k \|\nabla^i T\|_{L^q(\nu_e)}^q \right)^{1/q}.$$

Also we have ∇ maps $\mathbb{D}^{k,q}(\otimes^p T^* L_e(G), \nu_e)$ to $\mathbb{D}^{k-1,q}(\otimes^{p+1} T^* L_e(G), \nu_e)$. Let X_{h_1}, X_{h_2} be the vector field corresponding to $h_i \in H_0$. Then an easy calculation shows that $[X_{h_1}, X_{h_2}]F := X_{h_1}(X_{h_2}F) - X_{h_2}(X_{h_1}F)$ is equal to $X_{[h_2, h_1]}F$ for any smooth cylindrical function F . Here $[h_2, h_1](t) := [h_2(t), h_1(t)]$. Thus the exterior differential operator d is well-defined. We refer the reader to [2, 11] for the notion of tensor fields, covariant derivatives and Sobolev spaces on $L_e(G)$. We introduce a submanifold which is isomorphic to $L_e(G)$ by the solution of the stochastic differential equation in the sense of Stratonovich on G :

$$\begin{aligned} dX(t, a, w) &= (L_{X(t, a, w)})_* \circ dw_t, \\ X(0, a, w) &= a \in G. \end{aligned} \tag{2.1}$$

Here $L_a b = ab$ for $a, b \in G$ and w_t is the d -dimensional standard Brownian motion on $\mathbb{R}^d \cong \mathfrak{g}$ whose starting point is 0. That is, $w = (w_t) \in W^d$. We fix an ∞ -quasi-continuous version of $X(t, e, w)$ which is defined on a subset Ω of W^d . See Theorem 3.1 and Proposition 3.7. Let

$$S = \{w \in \Omega \mid X(1, e, w) = e\}. \tag{2.2}$$

There exists a probability measure μ_e on S which is given by

$$d\mu_e(w) = p(1, e, e)^{-1} \delta_e(X(1, e, w)) d\mu(w)$$

where $\delta_e(X(1, e, w))$ is a positive generalized Wiener function [37]. Note that μ_e has no mass on any Borel measurable subset A with $C_q^s(A) = 0$, where C_q^s denotes the (q, s) -capacity of A and

q (the parameter of integrability) is any number which is greater than 1 and s (the parameter of differentiability) is a sufficiently large positive number which depends on the dimension of G . Recall that a function f on W^d is said to be (q, s) -quasi-continuous if for any $\varepsilon > 0$, there exists a Borel measurable subset A_ε of W^d such that $C_q^s(A_\varepsilon) < \varepsilon$ and $f|_{A_\varepsilon}$ is continuous with respect to the topology of W^d . Hence, for sufficiently large s , (q, s) -quasi-continuous function is a μ_e -almost everywhere defined Borel measurable function. Also f is said to be ∞ -quasi-continuous when f is (q, s) -quasi-continuous for all (q, s) . We refer the reader to [37, 31, 22] for these notions and results. It is well-known that $X_*\mu_e = \nu_e$. In fact, the map $X : (S, \mu_e) \rightarrow (L_e(G), \nu_e)$ is isomorphism in the sense of Proposition 7.1. The covariant derivative ∇_S and the exterior differential operator d_S is defined on S using the H -derivative on W^d as in finite dimensions. These differential operators are defined on Sobolev spaces of covariant tensor fields $\mathbb{D}^{k,q}(\otimes^p T^*S)$ and the space of p -forms $\mathbb{D}^{k,q}(\wedge^p T^*S)$. We denote by $\|\cdot\|_{k,q}$ the Sobolev norm. See [23, 1] for these notions. Here we present a first main theorem which shows that any closed 1-form is exact on S .

Theorem 2.1. *Let G be a simply connected compact Lie group. There exists a sequence of ∞ -quasi-continuous functions $\rho_n \in \mathbb{D}^\infty(W^d)$ ($n \in \mathbb{N}$) for which the following statements hold.*

- (1) *For any n, w , $0 \leq \rho_n(w) \leq 1$ holds. Moreover for any $r > 1$, $k \in \mathbb{N}$, $\lim_{n \rightarrow \infty} C_r^k(\{w \in W^d \mid \rho_n(w) = 1\}^c) = 0$ and $\lim_{n \rightarrow \infty} \|\rho_n - 1\|_{r,k} = 0$.*
- (2) *Let $1 < p < 2$. Let $\theta \in L^2(\wedge^1 T^*S, d\mu_e) \cap \mathbb{D}^{\infty,p}(\wedge^1 T^*S, d\mu_e)$ and assume that $d_S \theta = 0$ μ_e -a.s. on S . Let $1 < q < p$ and k be a sufficiently large positive integer. Then there exist f and f_n which satisfy (i)-(v) below.*

- (i) *The function f is a μ_e -almost everywhere defined measurable function on S . Also f_n is a (q, k) -quasi-continuous function on W^d and $f_n \in \mathbb{D}^{k,q}(W^d)$.*
- (ii) *For any n , $f_n(w) = f(w)$ μ_e -almost everywhere on $\{\rho_n(w) \neq 0\} \cap S$ and $d_S f_n$ is equal to θ for μ_e -almost all elements of $\{\rho_n(w) \neq 0\} \cap S$.*
- (iii) *Let $\eta \in \mathbb{D}^\infty(W^d)$ be an ∞ -quasi-continuous function. Then it holds that $f \rho_n \eta \in L^1(S, \mu_e)$.*
- (iv) *For any n and ∞ -quasi-continuous map $\eta \in \mathbb{D}^\infty(W^d, H^*)$,*

$$\begin{aligned} & \int_S f(w) \rho_n(w) \left(-(d_S \rho_n(w), \eta(w)) + \rho_n(w) \widetilde{d_S^* \eta}(w) \right) d\mu_e(w) \\ &= \int_S \left(\tilde{\theta}(w) \rho_n(w) + f(w) d_S \rho_n(w), \rho_n(w) \eta(w) \right) d\mu_e(w), \end{aligned}$$

where $\widetilde{d_S^* \eta}$ is an ∞ -quasi-continuous modification of $d_S^* \eta$ and so on.

- (v) *Let $K > 0$ and ψ_K be a smooth function on \mathbb{R} such that $\psi_K(u) = u$ ($|u| \leq K$), $\psi_K(u) = -K - 1$ ($u \leq -K - 1$), $\psi_K(u) = K + 1$ ($u \geq K + 1$) and set $f^K = \psi_K(f)$. Then $f^K \in \mathbb{D}^{1,2}(S, \mu_e)$ and $d_S f^K = \psi'_K(f) \theta$ holds.*

The theorem above says that f is differentiable and $d_S f = \theta$ holds on S in the theorem's sense. The function ρ_n can be chosen independent of θ and actually they can be given more explicitly using the iterated integrals of the Brownian motion w . On $L_e(G)$, we can state a corresponding theorem to the above in a very simple form.

Theorem 2.2. *Let $1 < p < 2$. Let $\alpha \in L^2(\wedge^1 T^* L_e(G), \nu_e) \cap \mathbb{D}^{\infty, p}(\wedge^1 T^* L_e(G), \nu_e)$ and assume that $d\alpha = 0$ on $L_e(G)$. Then there exists a measurable function f on $L_e(G)$ such that the following hold.*

- (1) *Let ψ_K be the function which is defined in Theorem 2.1. Set $f^K = \psi_K(f)$. Then $f^K \in \mathbb{D}^{1,2}(L_e(G), \nu_e)$ and $df^K = \psi'_K(f)\alpha$.*
- (2) *For any $h \in H_0$ and $\varepsilon \geq 0$, we have*

$$f(e^{\varepsilon h} \gamma) - f(\gamma) = \int_0^\varepsilon \left(\alpha(e^{sh} \gamma), h \right) ds \quad \nu_e\text{-almost all } \gamma. \quad (2.3)$$

- (3) *For any $h \in H_0$ and $q < p$,*

$$\lim_{\varepsilon \rightarrow 0} \left\| \frac{f(e^{\varepsilon h} \gamma) - f(\gamma)}{\varepsilon} - (\alpha(\gamma), h) \right\|_{L^q(L_e(G), \nu_e)} = 0. \quad (2.4)$$

Using the above results, we have a vanishing theorem for the Hodge-Kodaira operator acting on 1-forms. First we give the definition of the Hodge-Kodaira operator.

Definition 2.3. *Let d be the exterior differential operator acting on 1-forms on $L_e(G)$. Let d^* be the adjoint operator of d . We consider the closable form on $L^2(\wedge^1 T^* L_e(G), \nu_e)$.*

$$\mathcal{E}(\alpha, \alpha) = (d\alpha, d\alpha)_{L^2(\wedge^2 T^* L_e(G))} + (d^* \alpha, d^* \alpha)_{L^2(L_e(G))},$$

which is defined on $\mathfrak{F}C_b^\infty(\wedge^1 T^ L_e(G))$. The Hodge-Kodaira operator \square acting on 1-forms is the non-negative generator of the closed form of the closure of the above.*

We note that $(dd^* + d^*d, \mathfrak{F}C_b^\infty(\wedge^1 T^* L_e(G)))$ is essentially self-adjoint. See [35]. The statement in [35] is concerning Hodge-Kodaira operators on submanifolds in Wiener spaces. However it can be applied to the case of $L_e(G)$ noting Proposition 7.1. The following is our vanishing theorem.

Theorem 2.4. *Let G be a simply connected compact Lie group. Then $\ker \square = \{0\}$. Also it holds that*

$$L^2(\wedge^1 T^* L_e(G)) = \overline{\{df \mid f \in \mathfrak{F}C_b^\infty(L_e(G))\}} \oplus \overline{\{d^* \alpha \mid \alpha \in \mathfrak{F}C_b^\infty(\wedge^2 T^* L_e(G))\}}. \quad (2.5)$$

Finally, we make further remarks.

- (1) As noted in the introduction, there are some difficulties to define a de Rham complex of differential forms in the Sobolev space category on the general path spaces $P_x(M)$, $L_x(M)$. However, we can define them on submanifolds in Wiener spaces. See [23, 24, 1, 5]. The proof in this paper can be applied to prove the vanishing of the 1-dimensional L^2 cohomology of the submanifold which is isomorphic to $L_x(M)$ in the case where $\pi_2(M) = 0$ which is equivalent to $\pi_1(L_x(M)) = 0$.

- (2) We mention the works of Kusuoka in the introduction. We explain Kusuoka's results. Kusuoka defined a local Sobolev spaces $\mathcal{D}_{loc}^{\infty, q}(U, d\mu)$ where U is a subset of W^d and q is the index of the integrability. Based on these Sobolev spaces and several results on the capacity which he introduced, Kusuoka announced the following theorems in [24]. Let M be a compact

Riemannian manifold which is isometrically embedded in \mathbb{R}^d . Let $P(x) : \mathbb{R}^d \rightarrow T_x M$ be the projection operator and consider a stochastic differential equation:

$$\begin{aligned} dX(t, x, w) &= P(X(t, x, w)) \circ dw_t, \\ X(0, x, w) &= x \in M. \end{aligned}$$

There exists a probability measure $d\mu_x = p(1, x, x)^{-1} \delta_x(X(1, x, w)) d\mu$ on the submanifold:

$$S = \{w \in W^d \mid X(1, x, w) = x\} \subset W^d.$$

Kusuoka proved that

Theorem 2.5. *There exists an isomorphism:*

$$\left\{ \alpha \in \mathcal{D}_{loc}^{\infty, q}(\wedge^p T^* S) \mid d_S \alpha = 0 \right\} / \left\{ d_S \beta \mid \mathcal{D}_{loc}^{\infty, q}(\wedge^{p-1} T^* S) \right\} \simeq H^p(\mathcal{M}_x, \mathbb{R}),$$

where

$$\begin{aligned} \mathcal{M}_x &= \left\{ h \in H \mid \xi(1, x, h) = x, \text{ where } \xi(t, x, h) \text{ is the solution to} \right. \\ &\quad \left. \dot{\xi}(t, x, h) = P(\xi(t, x, h)) \dot{h}(t), \xi(0, x, h) = x, t \geq 0 \right\} \end{aligned}$$

and $H^p(\mathcal{M}_x, \mathbb{R})$ is the de Rham cohomology of \mathcal{M}_x .

The subset \mathcal{M}_x is a Hilbert manifold in usual sense. Let $H^1 \cap L_x(M)$ be the subset of H^1 -paths of $L_x(M)$. Noting that $H^1 \cap L_x(M)$ and \mathcal{M}_x is C^∞ -homotopy equivalent, the conclusion of Theorem 2.5 is natural. Let $\square = d_S^* d_S + d_S d_S^*$ and $\square|_p$ be the restriction on p -forms. They are defined as the Friedrichs extension of them on some cores. Another Kusuoka's result is as follows.

Theorem 2.6. *There exists a mapping $j_p : \ker \square|_p \rightarrow H^p(\mathcal{M}_x, \mathbb{R})$ such that*

- (1) j_p is surjective for $p = 0, 1, 2, \dots$
- (2) j_p is injective for $p = 0, 1$.

Therefore our results give another proof to some special cases of his results. We may prove a vanishing theorem on a “contractible domain” of S using the method in our paper. Moreover, combining the usage of the Čech cohomology, we may prove the isomorphism between $H_1(H^1 \cap L_x(M), \mathbb{R})$ and $\ker \square|_1$ based on our proof. However we do not pursue this direction in this paper.

3 Preliminary from rough path analysis

The solutions of Itô's stochastic differential equations are measurable functions on W^d , but, they are not continuous in the uniform convergence topology of W^d in general. The reason of the discontinuity is clarified by the rough path analysis [29, 30, 15]. In rough path analysis, we need to consider objects which consist of the path and the iterated integrals. To explain the iterated

integrals, we take two continuous paths $x = x_t = (x_t^1, \dots, x_t^d)$, $y = y_t = (y_t^1, \dots, y_t^d)$ ($0 \leq t \leq 1$) on \mathbb{R}^d . Suppose that x or y is a bounded variation path. Then we can define for $0 \leq s \leq t \leq 1$

$$\begin{aligned} C(x, y)_{s,t} &= \int_s^t (x_u - x_s) \otimes dy_u \\ &= \sum_{1 \leq i, j \leq d} \left(\int_s^t (x_u^i - x_s^i) dy_u^j \right) e_i \otimes e_j \in \mathbb{R}^d \otimes \mathbb{R}^d \end{aligned} \quad (3.1)$$

as a Stieltjes integral. Here $e_i = {}^t(0, \dots, 1, \dots, 0)$. We introduce a function spaces for these iterated integrals. Let $\Delta = \{(s, t) \in \mathbb{R}^2 \mid 0 \leq s \leq t \leq 1\}$. Let V be a normed linear space. For a Borel measurable mapping $\phi : \Delta \rightarrow V$, define

$$\|\phi\|_{m,\theta} = \left\{ \int_0^1 \int_0^t \frac{|\phi(s, t)|^m}{(t-s)^{2+m\theta}} ds dt \right\}^{1/m},$$

where, m is a positive even integer and $0 < \theta < 1$. We denote the set of all measurable mappings ϕ from Δ to V satisfying $\|\phi\|_{m,\theta} < \infty$ by $L_{m,\theta}(\Delta \rightarrow V)$. Also we define $W_{m,\theta}(\Delta \rightarrow V) = L_{m,\theta}(\Delta \rightarrow V) \cap C(\Delta \rightarrow V)$, where $C(\Delta \rightarrow V)$ is the set of all continuous mappings from Δ to V . Note that $L_{m,\theta}(\Delta \rightarrow V)$ is a separable Banach space. Also for a measurable mapping $\phi : \Delta \rightarrow V$, define

$$\|\phi\|_{H,\theta} = \sup_{0 \leq s < t \leq 1} \frac{|\phi(s, t)|}{|t-s|^\theta}.$$

For $w \in W^d$, define $\bar{w}_{s,t} = w_t - w_s$ ($(s, t) \in \Delta$). We denote by $W_{m,\theta}(\mathbb{R}^d)$ all $w \in W^d$ with $\|\bar{w}\|_{m,\theta} < \infty$. We write $\|w\|_{m,\theta}$ instead of $\|\bar{w}\|_{m,\theta}$. Note that the Hölder norm $\|w\|_{H,\theta} := \|\bar{w}\|_{H,\theta}$ is weaker than the norm of $\|\cdot\|_{m,\theta}$ by a result of [16]. However this kind of statement does not hold for general $\phi \in W_{m,\theta}(\Delta \rightarrow V)$ without additional assumptions. See Lemma 3.5. Let $M_{m,\theta} = \sup_{x \neq 0, x \in W_{m,\theta/2}(\mathbb{R})} \frac{\|x\|_{H,\theta/2}}{\|x\|_{m,\theta/2}}$. Wiener measure μ satisfies that $\mu(W_{m,\theta/2}(\mathbb{R}^d)) = 1$ for all $0 < \theta < 1$. Note that $W_{m,\theta}(\mathbb{R}^d)$ is a separable Banach space. If x and y are Lipschitz continuous paths, then $C(x, y) \in W_{m,\theta}(\Delta \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d)$ for all (m, θ) with $m(1-\theta) > 2$. See Lemma 3.4.

Let $w = w_t = (w_t^1, \dots, w_t^d) \in W^d$ and $w(N)_t$ be the dyadic polygonal approximation of w . Namely, $w(N)_t = w_t$ for $t = \frac{k}{2^N}$ ($k = 0, 1, \dots, 2^N$) and $t \mapsto w(N)_t$ ($\frac{k}{2^N} \leq t \leq \frac{k+1}{2^N}$, $0 \leq k \leq 2^N - 1$) are linear functions. Also let $w(N)^i = (w(N), e_i)$ and define $w(N)^{\perp,i} = w^i - w(N)^i$, $w(N)^\perp = w - w(N)$. We need a probabilistic argument to define the integrals $C(w^i, w^j)_{s,t}$, $C(w, w)_{s,t}$ in contrast with $C(w(N), w)$, $C(w(N)^i, w^j)$. Indeed, they are Stratonovich integrals and we fix a version of them below.

Theorem 3.1. *Let Ω be the subset of W^d which consists of w satisfying the following (i)-(iii).*

- (i) $\lim_{N \rightarrow \infty} w(N)$ converges in $W_{m,\theta}(\mathbb{R}^d)$ for all (m, θ) with $m(1-\theta) > 2$.
- (ii) $\lim_{N \rightarrow \infty} C(w(N), w(N))$ converges in $W_{m,\theta}(\Delta \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d)$ for all (m, θ) with $m(1-\theta) > 2$. Moreover these converge with respect to all norms $\|\cdot\|_{H,\theta}$ ($0 < \theta < 1$).
- (iii) $\lim_{N \rightarrow \infty} C(w(N)^\perp, w(N))$ and $\lim_{N \rightarrow \infty} C(w(N), w(N)^\perp)$ converge to 0 in $W_{m,\theta}(\Delta \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d)$ for all (m, θ) with $m(1-\theta) > 2$. Moreover these converge to 0 with respect to all norms $\|\cdot\|_{H,\theta}$ ($0 < \theta < 1$).

Then Ω^c is a slim set and it holds that $H \subset \Omega$ and $\Omega + H \subset \Omega$.

A subset A of W^d is called a slim set if $C_q^s(A) = 0$ for all $s > 0$ and $q > 1$. See [31]. We note that $C(w^i, z^j)$ is meaningless even if both $w = (w^i)$ and $z = (z^j)$ belong to Ω generally. In rough path analysis, it is proved in many papers that the Wiener measure of the total set of paths which satisfy (i), (ii) above is 1. We need the property (iii) for our applications. The property (iii) is essential in [4] also. The fact that Ω^c is a slim set is proved in [19]. We give the proof of Theorem 3.1 for the sake of completeness, together with that of Theorem 3.2.

We use the following notation. For $w \in \Omega$, we define

$$C(w, w)_{s,t} = \lim_{N \rightarrow \infty} C(w(N), w(N))_{s,t} \quad (3.2)$$

$$C(w^i, w^j)_{s,t} = \lim_{N \rightarrow \infty} C(w(N)^i, w(N)^j)_{s,t} \quad (3.3)$$

where $1 \leq i, j \leq d$. Then it holds that for any $w = (w^i) \in \Omega$ and $0 \leq s \leq t \leq 1$,

$$C(w^i, w^j)_{s,t} = (w_t^i - w_s^i)(w_t^j - w_s^j) - C(w^j, w^i)_{s,t} \quad (3.4)$$

and $\|C(w(N)^{\perp, i}, w(N)^{\perp, j})\|_{m, \theta}$ converges to 0 for all $1 \leq i, j \leq d$ and (m, θ) with $m(1 - \theta) > 2$. For later use, we define $\Omega_N = \{w(N) \mid w \in \Omega\}$ and $\Omega_N^\perp = \{w - w(N) \mid w \in \Omega\}$. We denote the laws of $w(N)$ and $w(N)^\perp$ by μ_N and μ_N^\perp respectively. Note that Ω_N is the same as the set of all piecewise linear continuous paths w such that $t \mapsto w_t$ ($\frac{k}{2^N} \leq t \leq \frac{k+1}{2^N}, 0 \leq k \leq 2^N - 1$) is a linear function and this space is isomorphic to $\mathbb{R}^{2^N d}$. Also $w \in \Omega_N^\perp$ is equivalent to $w \in \Omega$ and $w(k/2^N) = 0$ for all integers with $0 \leq k \leq 2^N$. For simplicity, we may use the notation $\xi = (\xi^1, \dots, \xi^d)$ and $\eta = (\eta^1, \dots, \eta^d)$ to denote the element of Ω_N and Ω_N^\perp respectively.

Theorem 3.2. *Let us fix a positive even integer m and a positive number θ with $m(1 - \theta) > 2$. Let \mathfrak{T} be the weakest topology such that $w \in W^d \mapsto w(k/2^N)$ are continuous mappings for all k, N . The mappings $w \in \Omega \mapsto C(w^i, w^j) \in W_{m, \theta}(\Delta \rightarrow \mathbb{R})$ and $w \in \Omega \mapsto w \in W_{m, \theta/2}$ are ∞ -quasi-continuous for all i, j with respect to the topology \mathfrak{T} .*

To prove these theorems, we use the following lemmas.

Lemma 3.3. *Let $u \in \mathbb{D}^{s, q}(W^d)$ and \tilde{u} be the (q, s) -quasi-continuous version of u . Then there exists a positive number $C_{s, q}$ which is independent of u such that for all $R > 0$, the (q, s) -capacity satisfies*

$$C_q^s(\{w \in W^d \mid |\tilde{u}(w)| > R\}) \leq R^{-1} C_{s, q} \|u\|_{s, q}.$$

We refer the proof of Lemma 3.3 to [31]. In Lemma 3.4 (2), the estimates (3.6), (3.7), (3.8) hold with different constants under the weaker assumption $m(1 - \theta) > 2$. This is checked by the same proof as given below. Under the stronger assumption $m(1 - \theta) > 4$, the constants in the estimates (3.6), (3.7), (3.8) are simpler. We use this lemma in the proof of Lemma 5.2 too and the simpleness of the constants make the calculation simpler. Therefore we consider the stronger assumption. In the calculation below, constants C may change line by line.

Lemma 3.4. (1) *Let $x, y \in W_{m, \theta/2}(\mathbb{R})$ and set $(\bar{x} \cdot \bar{y})_{s,t} = (x_t - x_s)(y_t - y_s)$ ($0 \leq s \leq t \leq 1$). Then*

$$\|\bar{x} \cdot \bar{y}\|_{m, \theta} \leq M_{m, \theta} \|x\|_{m, \theta/2} \|y\|_{m, \theta/2}, \quad (3.5)$$

where $M_{m,\theta} = \sup_{x \neq 0, x \in W_{m,\theta/2}(\mathbb{R})} \frac{\|x\|_{H,\theta/2}}{\|x\|_{m,\theta/2}}$.

(2) Let $w \in W_{m,\theta/2}(\mathbb{R})$ and $\varphi \in H$. Suppose that $m(1-\theta) > 4$. Then

$$\|\varphi\|_{m,\theta/2} \leq \|\varphi\|_H, \quad (3.6)$$

$$\|C(w, \varphi)\|_{m,\theta} \leq \|w\|_{m,\theta/2} \|\varphi\|_H, \quad (3.7)$$

$$\|C(\varphi, w)\|_{m,\theta} \leq 2\|w\|_{m,\theta/2} \|\varphi\|_H, \quad (3.8)$$

$$\|D\|C(w, \varphi)\|_{m,\theta}^m \|H \leq C_{m,\theta} \|C(w, \varphi)\|_{m,\theta}^{m-1} \|\varphi\|_{m,\theta/2}, \quad (3.9)$$

$$\|D\|C(\varphi, w)\|_{m,\theta}^m \|H \leq C_{m,\theta} \|C(\varphi, w)\|_{m,\theta}^{m-1} \|\varphi\|_{m,\theta/2}, \quad (3.10)$$

where D denotes the H -derivative and $\|\cdot\|_H$ stands for the norm of the Cameron-Martin subspace H .

Proof. (1) We have

$$\begin{aligned} \|\bar{x} \cdot \bar{y}\|_{m,\theta}^m &= \int_0^1 \int_0^t \frac{|(x_t - x_s)(y_t - y_s)|^m}{(t-s)^{2+m\theta}} ds dt \\ &\leq \int_0^1 \int_0^t \frac{|(x_t - x_s)|^m (M_{m,\theta} \|y\|_{m,\theta/2})^m}{(t-s)^{2+m\theta/2}} ds dt = M_{m,\theta}^m \|x\|_{m,\theta/2}^m \|y\|_{m,\theta/2}^m. \end{aligned}$$

(2) The estimate (3.6) follows from

$$|\varphi_t - \varphi_s| \leq \|\varphi\|_H (t-s)^{1/2}. \quad (3.11)$$

We prove (3.7). Using the Hölder inequality, we have

$$\begin{aligned} &\frac{\left| \int_s^t (w(u) - w(s)) \dot{\varphi}(u) du \right|^m}{(t-s)^{2+m\theta}} \\ &\leq \frac{1}{(t-s)^{m\theta/2}} \left(\int_s^t \frac{|w(u) - w(s)|}{|u-s|^{(2+m\theta/2)/m}} |\dot{\varphi}(u)| du \right)^m \\ &\leq \int_s^t \frac{|w(u) - w(s)|^m}{|u-s|^{2+m\theta/2}} du \frac{1}{(t-s)^{m\theta/2}} \left(\int_s^t |\dot{\varphi}(u)|^{m/(m-1)} du \right)^{m-1}, \\ &\frac{1}{(t-s)^{m\theta/2}} \left(\int_s^t |\dot{\varphi}(u)|^{m/(m-1)} du \right)^{m-1} \leq \frac{1}{(t-s)^{m\theta/2}} \left(\int_s^t |\dot{\varphi}(u)|^2 du \right)^{m/2} (t-s)^{\frac{m-2}{2}} \\ &\leq (t-s)^{\frac{(m-2)-m\theta}{2}} \|\varphi\|_H^m \\ &\leq \|\varphi\|_H^m. \end{aligned}$$

Hence

$$\begin{aligned} \|C(w, \varphi)\|_{m,\theta}^m &\leq \int_0^1 \int_0^t \left(\int_s^t \frac{|w(u) - w(s)|^m}{|u-s|^{2+m\theta/2}} du \right) ds dt \|\varphi\|_H^m \\ &= \int_0^1 \int_0^t \left(\int_0^u \frac{|w(u) - w(s)|^m}{|u-s|^{2+m\theta/2}} ds \right) du dt \|\varphi\|_H^m \\ &\leq \|w\|_{m,\theta/2}^m \|\varphi\|_H^m. \end{aligned}$$

We prove (3.8). Noting that for 1-dimensional paths x, y ,

$$C(x, y)_{s,t} = (x_t - x_s)(y_t - y_s) - C(y, x)_{s,t}, \quad (3.12)$$

we have

$$\begin{aligned} \|C(\varphi, w)\|_{m,\theta} &\leq \|C(w, \varphi)\|_{m,\theta} + \left(\int_0^1 \int_0^t \frac{|(w_t - w_s)(\varphi_t - \varphi_s)|^m}{(t-s)^{2+m\theta}} ds dt \right)^{1/m} \\ &\leq \|C(w, \varphi)\|_{m,\theta} + \|w\|_{m,\theta/2} \|\varphi\|_H, \end{aligned}$$

where we have used (3.11). This and (3.7) prove (3.8). We consider (3.9). Let $h \in H$. We have

$$D_h \left(\int_s^t (w_u - w_s) d\varphi_u \right) = (\varphi_t - \varphi_s)(h_t - h_s) - \int_s^t (\varphi_u - \varphi_s) \dot{h}_u du.$$

Therefore

$$\begin{aligned} D_h (\|C(w, \varphi)\|_{m,\theta}^m) &= m \int_0^1 \int_0^t \frac{((\varphi_t - \varphi_s)(h_t - h_s) - C(\varphi, h)_{s,t}) C(w, \varphi)_{s,t}^{m-1}}{(t-s)^{2+m\theta}} ds dt. \end{aligned}$$

Using the Hölder inequality, (3.7) and (3.11), we get

$$\begin{aligned} D_h (\|C(w, \varphi)\|_{m,\theta}^m) &\leq C_{m,\theta} (\|\varphi\|_{m,\theta/2} \|h\|_H + \|C(\varphi, h)\|_{m,\theta}) \|C(w, \varphi)\|_{m,\theta}^{m-1} \\ &\leq 2C_{m,\theta} \|\varphi\|_{m,\theta/2} \|h\|_H \|C(w, \varphi)\|_{m,\theta}^{m-1} \end{aligned}$$

which proves (3.9). As for (3.10), noting that

$$D_h \left(\int_s^t (\varphi_u - \varphi_s) dw_u \right) = \int_s^t (\varphi_u - \varphi_s) \dot{h}_u du,$$

we can prove (3.10) similarly to (3.9). \square

Lemma 3.5. *Let $0 < \theta < 1$ and m be a positive even integer. There exists a positive constant $N_{m,\theta}$ such that for all $x, y \in H$, we have*

$$\|C(x, y)\|_{H,\theta} \leq N_{m,\theta} (\|C(x, y)\|_{m,\theta} + \|x\|_{m,\theta/2} \|y\|_{m,\theta/2}). \quad (3.13)$$

Proof. It suffices to prove the case where $\|y\|_{m,\theta/2} = 1$. In this case, the proof is almost similar to [16] noting Chen's identity: $C(x, y)_{s,t} = C(x, y)_{s,r} + C(x, y)_{r,t} + (x(r) - x(s)) \otimes (y(t) - y(r))$ $0 < s < r < t < 1$. See also [14] \square

Proof of Theorem 3.1 and Theorem 3.2. Let $z(N) = w(N) - w(N-1)$ ($N = 1, 2, \dots$), where $w(0) = 0$. Then $\{z(N); N = 1, 2, \dots\}$ are independent random variables with values in the set of piecewise linear functions. Using explicit form of $z(N)$, we have

$$E[|w(N)_t - w(N)_s|^2] \leq d|t - s| \quad (3.14)$$

$$E[|z(N)_t - z(N)_s|^2] \leq C_d \min(|t - s|, 2^{-N}) \quad (3.15)$$

$$E[|w(N)_t^\perp - w(N)_s^\perp|^2] \leq C_d \min(|t - s|, 2^{-N}). \quad (3.16)$$

We estimate L^2 -norm of $\|z(N)^i\|_{m,\theta/2}^m$.

$$\begin{aligned}
& \left\| \|z(N)^i\|_{m,\theta/2}^m \right\|_{L^2(\mu)} \\
&= \left\{ \int_{W^d} d\mu \iint_{(s,t) \in \Delta, (s',t') \in \Delta} \frac{(z(N)_t^i - z(N)_s^i)^m (z(N)_{t'}^i - z(N)_{s'}^i)^m}{|t-s|^{2+m\theta/2} |t'-s'|^{2+m\theta/2}} ds dt ds' dt' \right\}^{1/2} \\
&\leq \iint_{(s,t) \in \Delta} \frac{E[(z(N)_t^i - z(N)_s^i)^{2m}]^{1/2}}{|t-s|^{2+m\theta/2}} ds dt \\
&= C_m \iint_{(s,t) \in \Delta} \frac{E[(z(N)_t^i - z(N)_s^i)^2]^{m/2}}{|t-s|^{2+m\theta/2}} ds dt \\
&\leq C_m \iint_{(s,t) \in \Delta} \frac{\min(|t-s|, 2^{-N})^{m/2}}{|t-s|^{2+m\theta/2}} ds dt \\
&\leq C_m \iint_{(s,t) \in \Delta} |t-s|^{\frac{m}{2}(1-\varepsilon-\theta)-2} 2^{-\varepsilon m N/2} ds dt.
\end{aligned}$$

Thus if $m(1-\theta) > 2$, choosing an appropriate $\varepsilon > 0$, there exists a positive number $C_{m,\theta,\varepsilon}$

$$\left\| \|z(N)^i\|_{m,\theta/2}^m \right\|_{L^2(\mu)} \leq C_{m,\theta,\varepsilon} 2^{-\varepsilon m N/2}. \quad (3.17)$$

Noting $E[|w(N)_t^i - w(N)_s^i|^{2m}] \leq E[|w_t^i - w_s^i|^{2m}] \leq C_m |t-s|^m$ and by the calculation similar to the above, if $m(1-\theta) > 2$,

$$\left\| \|w(N)\|_{m,\theta/2}^m \right\|_{L^2(\mu)} \leq C_{m,\theta} \quad (3.18)$$

$$\left\| \|w(N)^{\perp,i}\|_{m,\theta/2}^m \right\|_{L^2(\mu)} \leq C_{m,\theta,\varepsilon} 2^{-\varepsilon m N/2}. \quad (3.19)$$

Hence by (3.5),

$$\left\| \left\| \overline{w(N)^i} \cdot \overline{z(N+1)^j} \right\|_{m,\theta}^m \right\|_{L^2(\mu)} \leq C_{m,\theta,\varepsilon} 2^{-\varepsilon m(N+1)/2},$$

where $\left(\overline{w(N)^i} \cdot \overline{z(N+1)^j} \right)_{s,t} = (w(N)_t^i - w(N)_s^i) (z(N+1)_t^j - z(N+1)_s^j)$. Similarly,

$$\left\| \left\| \overline{w(N)^{\perp,i}} \cdot \overline{w(N)^j} \right\|_{m,\theta} \right\|_{L^2(\mu)} \leq C_{m,\theta,\varepsilon} 2^{-\varepsilon m(N+1)/2}.$$

We estimate $C(z(N+1)^i, w(N)^j)_{s,t}$. By the independence of $z(N+1)^i$ and $w(N)^j$,

$$\begin{aligned}
E[C(z(N+1)^i, w(N)^j)_{s,t}^m] &= C_m E \left[\left(\int_s^t (z(N+1)_u^i - z(N+1)_s^i)^2 du \right)^{m/2} \right] \\
&\leq C_m E \left[\int_s^t (z(N+1)_u^i - z(N+1)_s^i)^m du \right] \left(\int_s^t 1 du \right)^{(m-2)/2} \\
&\leq C_m \min(|t-s|^m, 2^{-(N+1)m/2}).
\end{aligned}$$

Using this,

$$\begin{aligned} \left\| \|C(z(N+1)^i, w(N)^j)\|_{m,\theta}^m \right\|_{L^2(\mu)} &\leq \iint_{(s,t) \in \Delta} \frac{E [C(z(N+1)^i, w(N)^j)_{s,t}^{2m}]^{1/2}}{|t-s|^{2+m\theta}} ds dt \\ &\leq 2^{-(N+1)m\varepsilon/2} \iint_{(s,t) \in \Delta} |t-s|^{m(1-\varepsilon-\theta)-2} ds dt. \end{aligned}$$

Hence if $m(1-\theta) > 1$, then we have

$$\left\| \|C(z(N+1)^i, w(N)^j)\|_{m,\theta}^m \right\|_{L^2(\mu)} \leq C_{m,\theta,\varepsilon} 2^{-(N+1)m\varepsilon/2}.$$

Similarly if $m(1-\theta) > 1$,

$$\begin{aligned} \left\| \|C(w(N)^{\perp,i}, w(N)^j)\|_{m,\theta}^m \right\|_{L^2(\mu)} &\leq C_{m,\theta,\varepsilon} 2^{-Nm\varepsilon/2}, \\ \left\| \|C(z(N)^i, z(N)^j)\|_{m,\theta}^m \right\|_{L^2(\mu)} &\leq C_{m,\theta,\varepsilon} 2^{-Nm\varepsilon/2} \quad (i \neq j). \end{aligned}$$

When $i = j$, under the assumption that $m(1-\theta) > 2$,

$$\begin{aligned} \left\| \|C(z(N)^i, z(N)^i)\|_{m,\theta}^m \right\|_{L^2(\mu)} &= \left(\frac{1}{2} \right)^m \left\| \|\overline{z(N)^i} \cdot \overline{z(N)^i}\|_{m,\theta}^m \right\|_{L^2(\mu)} \\ &\leq \left(\frac{M_{m,\theta}}{2} \right)^m \left\| \|z(N)^i\|_{m,\theta/2}^{2m} \right\|_{L^2(\mu)}. \end{aligned}$$

Let

$$\begin{aligned} A_{N,i} &= \left\{ w \mid \|z(N+1)^i\|_{m,\theta/2} > N^{-2} \right\}, \\ B_{N,i,j} &= \left\{ w \mid \|C(w(N+1)^i, w(N+1)^j) - C(w(N)^i, w(N)^j)\|_{m,\theta} > N^{-2} \right\}, \\ C_{N,i,j} &= \left\{ w \mid \|\overline{w(N)^{\perp,i}} \cdot \overline{w(N)^j}\|_{m,\theta} > N^{-2} \right\} \\ D_{N,i,j} &= \left\{ w \mid \|C(w(N)^{\perp,i}, w(N)^j)\|_{m,\theta} > N^{-2} \right\}. \end{aligned}$$

Note that $\|z(N+1)^i\|_{m,\theta/2}^m$, $\|C(w(N+1)^i, w(N+1)^j) - C(w(N)^i, w(N)^j)\|_{m,\theta}^m$, $\|\overline{w(N)^{\perp,i}} \cdot \overline{w(N)^j}\|_{m,\theta}^m$, $\|C(w(N)^{\perp,i}, w(N)^j)\|_{m,\theta}^m$, are Wiener chaos of order at most $2m$. Hence by the hypercontractivity of the Ornstein-Uhlenbeck semi-group, their L^2 -norms and the (q, s) -Sobolev norms are equivalent for any $q \geq 2, s > 0$. By Lemma 3.3 and the above estimates, we obtain

$$\max(C_q^s(A_{N,i}), C_q^s(C_{N,i,j}), C_q^s(D_{N,i,j})) \leq C_{s,q,m,\theta,\varepsilon} N^{2m} 2^{-\varepsilon m N/2}. \quad (3.20)$$

Since

$$\begin{aligned} &C(w(N+1)^i, w(N+1)^j) - C(w(N)^i, w(N)^j) \\ &= (w(N)_t^i - w(N)_s^i) \left(z(N+1)_t^j - z(N+1)_s^j \right) - C(z(N+1)^j, w(N)^i)_{s,t} \\ &\quad + C(z(N+1)^i, w(N)^j)_{s,t} + C(z(N+1)^i, z(N+1)^j)_{s,t}, \end{aligned} \quad (3.21)$$

using the subadditivity of the capacity, we have

$$C_q^s(B_{N,i,j}) \leq C_{s,q,m,\theta,\varepsilon} N^{2m} 2^{-\varepsilon m N/2}. \quad (3.22)$$

Here we note that $A_{N,i}, B_{N,i,j}, C_{N,i,j}, D_{N,i,j}$ depend on (m, θ) satisfying $m(1 - \theta) > 2$. Let

$$E = \cup_{1 \leq i,j \leq d, m, \theta \in \mathbb{Q}} \left\{ (\limsup_{N \rightarrow \infty} A_{N,i}) \cup (\limsup_{N \rightarrow \infty} B_{N,i,j}) \cup (\limsup_{N \rightarrow \infty} C_{N,i,j}) \cup (\limsup_{N \rightarrow \infty} D_{N,i,j}) \right\}.$$

By (3.20) and (3.22), E is a slim set. Since $E^c \subset \Omega$, Ω^c is a slim set. The properties that $H \subset \Omega$ and $\Omega + H \subset \Omega$ follows from the estimates in Lemma 3.4. To complete the proof of Theorem 3.1, we need to show

- (a) the sequences of iterated integrals converge with respect to $\|\cdot\|_{H,\theta}$,
- (b) the limit is continuous with respect to $(s, t) \in \Delta$.

The item (a) follows from Lemma 3.5 and the convergences in $L_{m,\theta}$. The item (b) follows from

(a). Now we prove Theorem 3.2. Let $E_{K,m,\theta} = \cap_{1 \leq i,j \leq d} \left\{ \cap_{N=K}^{\infty} (A_{N,i}^c \cap B_{N,i,j}^c \cap C_{N,i,j}^c \cap D_{N,i,j}^c) \right\}$. Then $w(N), C(w(N), w(N))$ converges uniformly with respect to $\|\cdot\|_{m,\theta/2}$ on $E_{K,m,\theta}$. Therefore $C(w, w), w$ is continuous with respect to \mathfrak{T} on $E_{K,m,\theta} \cap \Omega$. For any (s, q) and $\varepsilon > 0$, we have $C_q^s(E_{K,m,\theta}^c) < \varepsilon$ for sufficiently large K . This completes the proof of Theorem 3.2. \square

We fix a version of the solution of SDE (2.1) using Theorem 3.1. To this end, we introduce a distance function on Ω .

Definition 3.6. Let $(2/3) < \theta < \theta' < 1$ and assume $m(1 - \theta') > 2$. For $w, z \in \Omega$, let

$$d_{\Omega}(w, z) = \max \left\{ \max_{i,j} \|C(w^i, w^j) - C(z^i, z^j)\|_{H,\theta}, \max_i \|w^i - z^i\|_{m,\theta'/2} \right\}. \quad (3.23)$$

We note that (Ω, d_{Ω}) is a separable metric space. For $h \in H$, let $X(t, a, h)$ be the solution to the following ODE:

$$\begin{aligned} \dot{X}(t, a, h) &= (L_{X(t,a,h)})_* \dot{h}_t, \\ X(0, a, h) &= a \in G. \end{aligned}$$

By the assumption that $\frac{2}{3} < \theta < 1$, the topology by the distance d_{Ω} is stronger than the p -variation topology with $p > \frac{2}{\theta}$. Hence by Theorem 3.1 and the universal limit theorem [29, 30, 15], for any $w \in \Omega$, $t \geq 0, a \in G$, the limit

$$\lim_{N \rightarrow \infty} X(t, a, w(N)) \quad (3.24)$$

exists. We denote the limit by $X(t, a, w)$. For this limit, we have the following.

Proposition 3.7. The measurable mapping $X : [0, \infty) \times G \times \Omega \rightarrow G$ satisfies the following.

- (1) $X(t, a, w)$ is a version of the solution to the SDE (2.1).
- (2) For any a , the mapping $w \mapsto X(\cdot, a, w) \in C([0, 1] \rightarrow G)$ is continuous in the sense that there exists an increasing function F on \mathbb{R} such that for all $w, z \in \Omega$,

$$\sup_{0 \leq t \leq 1} d(X(t, a, w), X(t, a, z)) \leq F(\max\{d_{\Omega}(0, w), d_{\Omega}(0, z)\}) d_{\Omega}(w, z).$$

Moreover the mapping $w \mapsto X(\cdot, a, w)$ is ∞ -quasi-continuous with respect to the supremum norm of W^d for any a .

(3) For all t, a, w , $X(t, a, w) = aX(t, e, w)$. In particular, the mapping $a \mapsto X(t, a, w)$ is a C^∞ -diffeomorphism.

(4) For any $\phi \in H^1([0, 1] \rightarrow G \mid \phi_0 = e)$, it holds that

$$X(t, \phi_t, w) = X(t, e, w + \zeta(\phi, w)), \quad (3.25)$$

where $\zeta(\phi, w)$ is the solution to

$$\dot{\zeta}(\phi, w)_t = \text{Ad}(X(t, e, w)^{-1}) \left(\phi_t^{-1} \dot{\phi}_t \right) \quad t > 0 \quad (3.26)$$

$$\zeta(\phi)_0 = 0. \quad (3.27)$$

(5) For $h \in H$, let $Z(t, h, w)$ be the H^1 -path on G which satisfies the ODE:

$$Z(t, h, w)^{-1} \dot{Z}(t, h, w) = \text{Ad}(X(t, e, w)) \dot{h}_t \quad t > 0 \quad (3.28)$$

$$Z(0, h, w) = e. \quad (3.29)$$

Then it holds that $X(t, Z(t, h, w), w) = X(t, e, w + h)$.

(6) For any $h \in H$

$$\zeta(Z(\cdot, h, w), w) = h. \quad (3.30)$$

Proof. Part (1) is a standard result in stochastic analysis. Part (2) is a consequence of rough path analysis. The claim that (3),(4),(5),(6) hold for almost all w is also standard in stochastic analysis. However, these identities hold for all $w \in \Omega$. This follows from the fact:

- (i) the claims (3),(4),(5),(6) hold for all $w \in H$,
- (ii) The Cameron-Martin subspace H is a dense subset in Ω with respect to the topology defined by d_Ω ,
- (iii) Part (2).

□

The following will be used in the next section.

Lemma 3.8. Suppose that $m(1 - \theta) > 2$. Let $(x, y) = (w(N)^i, w(N)^j), (w^i, w^j)$ for $i \neq j$ or $(x, y) = (w(N)^i, w(N)^{\perp, j}), (w(N)^{\perp, i}, w(N)^j)$ for any i, j . Then the following estimates hold for almost all w .

$$\|D^k \|x\|_{m, \theta/2}^m \|H\| \leq C_{m, \theta, k} \|x\|_{m, \theta/2}^{m-k} \quad \text{for all } 1 \leq k \leq m, \quad (3.31)$$

$$\begin{aligned} \|D^k \|C(x, y)\|_{m, \theta}^m \|H\| &\leq C_{m, \theta, k} \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} \left(\|x\|_{m, \theta/2}^2 + \|y\|_{m, \theta/2}^2 \right)^{(k-2l)/2} \|C(x, y)\|_{m, \theta}^{m+l-k}. \\ &\text{for all } 1 \leq k \leq 2m. \end{aligned} \quad (3.32)$$

Proof. We consider the case where $k = 1$ and $x = w(N)^i$ in (3.31). The proof of other cases are similar to it. We have

$$\begin{aligned} |D_h \|x\|_{m,\theta/2}^m| &= \left| m \int_0^1 \int_0^t \frac{(h(N)_t^i - h(N)_s^i)(w(N)_t^i - w(N)_s^i)^{m-1}}{(t-s)^{2+m\theta/2}} ds dt \right| \\ &= m \|h(N)^i\|_{m,\theta/2} \|w(N)^i\|_{m,\theta/2}^{m-1} \\ &\leq C_{m,\theta} \|h\|_H \|w(N)^i\|_{m,\theta/2}^{m-1} \end{aligned}$$

which implies (3.31). We prove (3.32) in the case where $k = 1$. Let $(x, y) = (w(N)^i, w(N)^j)$ ($i \neq j$). Then

$$\begin{aligned} &|D_h \|C(x, y)\|_{m,\theta}^m| \\ &= m \left| \int_0^1 \int_0^t \frac{(C(h(N)^i, w(N)^j)_{s,t} + C(w(N)^i, h(N)^j)_{s,t}) (C(x, y)_{s,t}^{m-1})}{(t-s)^{2+m\theta}} ds dt \right| \\ &\leq m (\|C(h(N)^i, w(N)^j)\|_{m,\theta} + \|C(w(N)^i, h(N)^j)\|_{m,\theta}) \|C(x, y)\|_{m,\theta}^{m-1} \\ &\leq C_{m,\theta} (\|w(N)^i\|_{m,\theta/2} \|h(N)^j\|_H + \|w(N)^j\|_{m,\theta/2} \|h(N)^i\|_H) \|C(x, y)\|_{m,\theta}^{m-1}, \end{aligned}$$

where we have applied Lemma 3.4 (2) in the case where $m(1-\theta) > 2$. This implies (3.32). We can check the other cases in similar ways. \square

4 A Poincaré's lemma on a certain domain in a Wiener space

The reader may find the following statement in Remark 3.2 in [4]. We apply this lemma to Dirichlet forms on open subsets in Euclidean spaces. For the sake of completeness, we give the proof.

Lemma 4.1. *Let (X, μ) and (Y, ν) be probability spaces. Let $dm = d\mu \otimes d\nu$. Assume that we are given Dirichlet forms $(\mathcal{E}_X, D(\mathcal{E}_X))$, $(\mathcal{E}_Y, D(\mathcal{E}_Y))$ on $L^2(X, \mu)$ and $L^2(Y, \nu)$. Moreover we assume that \mathcal{E}_X , \mathcal{E}_Y has the square field operators Γ_X and Γ_Y respectively. Let U be a measurable subset of $X \times Y$ with $m(U) > 0$. Let $U_x = \{y \in Y \mid (x, y) \in U\}$ and $U^y = \{x \in X \mid (x, y) \in U\}$. Let $A = \{x \in X \mid \nu(U_x) > 0\}$ and $B = \{y \in Y \mid \mu(U^y) > 0\}$. We assume that*

(1) *There exists $\tilde{A} \subset A$ such that $\mu(A \setminus \tilde{A}) = 0$ and $\delta = \inf_{x, x' \in \tilde{A}} \nu(U_x \cap U_{x'}) > 0$. Moreover there exists a positive number C_2 such that for any $x \in \tilde{A}$ and $g \in D(\mathcal{E}_Y)$,*

$$\text{Var}(g; U_x) \leq \frac{C_2}{\nu(U_x)} \int_{U_x} \Gamma_Y g(y) d\nu(y). \quad (4.1)$$

Here $\text{Var}(g; U_x)$ denotes the variance of g with respect to the probability measure $d\nu|_{U_x}/\nu(U_x)$. In the statement below too, we use Var in this sense.

(2) *There exists $\tilde{B} \subset B$ such that $\nu(B \setminus \tilde{B}) = 0$ and there exists a positive number C_1 such that for any $y \in \tilde{B}$ and $h \in D(\mathcal{E}_X)$*

$$\text{Var}(h; U^y) \leq \frac{C_1}{\mu(U^y)} \int_{U^y} \Gamma_X h(x) d\mu(x). \quad (4.2)$$

Let us denote $z = (x, y) \in X \times Y$. Then we have for $f = f(z) = f(x, y)$,

$$\text{Var}(f; U) \leq \frac{3}{\delta m(U)} \int_U \left(\frac{C_1}{m(U)} \Gamma_X f(x, y) + C_2 \Gamma_Y f(x, y) \right) dm(z). \quad (4.3)$$

Proof. Let $x, x' \in \tilde{A}$, $y \in U_x, y' \in U_{x'}, z \in U_x \cap U_{x'}$. Noting that

$$\begin{aligned} & (f(x, y) - f(x', y'))^2 \\ & \leq 3 \left\{ (f(x, y) - f(x, z))^2 + (f(x, z) - f(x', z))^2 + (f(x', z) - f(x', y'))^2 \right\}, \end{aligned} \quad (4.4)$$

and $\nu(U_x \cap U_{x'}) > \delta$, we have

$$\begin{aligned} (f(x, y) - f(x', y'))^2 & \leq \frac{3}{\delta} \int_{U_x \cap U_{x'}} (f(x, y) - f(x, z))^2 d\nu(z) \\ & \quad + \frac{3}{\delta} \int_{U_x \cap U_{x'}} (f(x, z) - f(x', z))^2 d\nu(z) \\ & \quad + \frac{3}{\delta} \int_{U_x \cap U_{x'}} (f(x', z) - f(x', y'))^2 d\nu(z) \\ & = I_1 + I_2 + I_3. \end{aligned} \quad (4.5)$$

We estimate I_i .

$$\begin{aligned} & \int_{x, x' \in \tilde{A}, y \in U_x, y' \in U_{x'}} I_1 d\mu(x) d\mu(x') d\nu(y) d\nu(y') \\ & \leq \frac{3}{\delta} \int_{x \in \tilde{A}, y, z \in U_x} (f(x, y) - f(x, z))^2 d\nu(y) d\nu(z) d\mu(x) m(U) \\ & \leq \frac{3C_2 m(U)}{\delta} \int_{x \in \tilde{A}, y \in U_x} 2\nu(U_x) \Gamma_Y f(x, y) d\nu(y) d\mu(x). \end{aligned} \quad (4.6)$$

$$\begin{aligned} & \int_{x, x' \in \tilde{A}, y \in U_x, y' \in U_{x'}} I_2 d\mu(x) d\mu(x') d\nu(y) d\nu(y') \\ & = \frac{3}{\delta} \int_{x, x' \in \tilde{A}} \left(\nu(U_x) \nu(U_{x'}) \int_{z \in U_x \cap U_{x'}} (f(x, z) - f(x', z))^2 d\nu(z) \right) d\mu(x) d\mu(x') \\ & \leq \frac{3}{\delta} \int_{x, x' \in \tilde{A} \cap U^z, z \in Y} \left\{ (f(x, z) - f(x', z))^2 d\mu(x) d\mu(x') \right\} d\nu(z) \\ & = \frac{3}{\delta} \int_{x, x' \in U^z, z \in \tilde{B}} \left\{ (f(x, z) - f(x', z))^2 d\mu(x) d\mu(x') \right\} d\nu(z) \\ & \leq \frac{3}{\delta} \int_{\tilde{B}} d\mu(z) 2C_1 \mu(U^z) \int_{U^z} \Gamma_X f(x, z) d\mu(x). \end{aligned} \quad (4.7)$$

As to I_3 , we have the same estimate for I_1 :

$$\begin{aligned} & \int_{x, x' \in \tilde{A}, y \in U_x, y' \in U_{x'}} I_3 d\mu(x) d\mu(x') d\nu(y) d\nu(y') \\ & \leq \frac{3C_2 m(U)}{\delta} \int_{x \in \tilde{A}, y \in U_x} 2\nu(U_x) \Gamma_Y f(x, y) d\nu(y) d\mu(x). \end{aligned} \quad (4.8)$$

Since

$$\begin{aligned} & \int_{x, x' \in \tilde{A}, y \in U_x, y' \in U_{x'}} (f(x, y) - f(x', y'))^2 d\mu(x) d\mu(x') d\nu(y) d\nu(y') \\ & = 2m(U) \int_U \left(f(z) - \frac{1}{m(U)} \int_U f(z) dm(z) \right)^2 dm(z), \end{aligned} \quad (4.9)$$

the above estimates complete the proof. \square

To apply the lemma above to $U_{r,\varphi}$ which we will define later, we need uniform positivity of probabilities of intersections of subsets of a Wiener space (Lemma 4.4 (1)). First we begin by the following.

Lemma 4.2. *Let us consider the case where $d = 1$. That is, w is a real valued continuous path. Let $0 < \theta < \theta' < 1$ and $m(1 - \theta) > 2$. Let $z^1, \dots, z^l \in W_{m,\theta/2}(\mathbb{R})$ and define*

$$U_N(z^1, \dots, z^l; \varepsilon) = \left\{ w \in \Omega \mid \max_{1 \leq i \leq l} \{ \|w(N)\|_{m,\theta'/2}, \|C(w(N), z^i)\|_{m,\theta}, \|C(z^i, w(N))\|_{m,\theta} \} < \varepsilon \right\},$$

where ε is a positive number. Then for fixed $l, r > 0$ and $\varepsilon > 0$, we have

$$\inf \left\{ \mu \left(U_N(z^1, \dots, z^l; \varepsilon) \right) \mid \max_{1 \leq i \leq l} \|z^i\|_{m,\theta'/2} \leq r, N \in \mathbb{N} \right\} > 0. \quad (4.10)$$

For later use, we denote the infimum in (4.10) by $C(l, \varepsilon, r, m, \theta, \theta')$.

To prove the lemma above, we need a lemma. Let x be a real-valued continuous function on $[0, 1]$ and w be the 1-dimensional Brownian motion. Then the stochastic integral (Wiener integral) $B(x, w)$ is defined for almost all w as continuous functions of $(s, t) \in \Delta$:

$$B(x, w)_{s,t} = \int_s^t (x_u - x_s) dw_u. \quad (4.11)$$

Also we set $B(w, x)_{s,t} = (\bar{x} \cdot \bar{w})_{s,t} - B(x, w)_{s,t}$. As for the notation $(\bar{x} \cdot \bar{w})_{s,t}$, see Lemma 3.4 (1). For these stochastic integrals, we have the following estimates.

Lemma 4.3. *Assume $m(1 - \theta) > 2$. Stochastic integrals $B(x, w), B(w, x)$ take values in $W_{m,\theta/2}$ for almost all w and*

$$E [\|B(x, w)\|_{m,\theta}^m + \|B(w, x)\|_{m,\theta}^m] \leq C_{m,\theta} \|x\|_{m,\theta/2}^m. \quad (4.12)$$

Also we have

$$\lim_{N \rightarrow \infty} E [\|C(x, w(N)) - B(x, w)\|_{m,\theta}^m + \|C(w(N), x) - B(w, x)\|_{m,\theta}^m] = 0. \quad (4.13)$$

Proof. We have

$$\begin{aligned} E \left[\int_0^1 \int_0^t \frac{B(x, w)_{s,t}^m}{|t-s|^{2+m\theta}} ds dt \right] &= C_m \int_0^1 \int_0^t \frac{\left(\int_s^t (x_u - x_s)^2 du \right)^{m/2}}{(t-s)^{2+m\theta}} ds dt \\ &\leq C_m \int_0^1 \int_0^t \frac{(t-s)^{\frac{m}{2}-1} \int_s^t (x_u - x_s)^m du}{(t-s)^{2+m\theta}} ds dt \\ &\leq C_m \int_0^1 \int_0^t \frac{\int_s^t (x_u - x_s)^m du}{(t-s)^{2+m\theta/2}} ds dt \\ &\leq C_m \|x\|_{m,\theta/2}^m. \end{aligned}$$

Noting that $B(w, x)_{s,t} = (w_t - w_s)(x_t - x_s) - B(x, w)_{s,t}$ and

$$E \left[\int_0^1 \int_0^t \frac{(w_t - w_s)^m}{(t-s)^{2+m\theta/2}} ds dt \right] < \infty$$

we complete the proof of (4.12). We prove (4.13). We have

$$\begin{aligned} & \left\| \|C(x, w(N)) - B(x, w)\|_{m,\theta}^m \right\|_{L^2(\mu)} \\ & \leq \iint_{\{(s,t) \in \Delta\}} \frac{E \left[(C(x, w(N))_{s,t} - B(x, w)_{s,t})^{2m} \right]^{1/2}}{(t-s)^{2+m\theta}} ds dt. \end{aligned} \quad (4.14)$$

Note that

$$E \left[(C(x, w(N))_{s,t} - B(x, w)_{s,t})^{2m} \right]^{1/2} \leq C_m \psi_N(s, t)$$

where $\psi_N(s, t) = E \left[(C(x, w(N))_{s,t} - B(x, w)_{s,t})^2 \right]^{m/2}$. Also

$$\psi_N(s, t) \leq E \left[B(x, w)_{s,t}^2 \right]^{m/2} =: \psi(s, t).$$

This follows from that $w - w(N)$ and $w(N)$ are independent. It holds that $\lim_{N \rightarrow \infty} \psi_N(s, t) = 0$ for all (s, t) and $\iint_{\Delta} \frac{\psi(s, t)}{(t-s)^{2+m\theta}} ds dt < \infty$. Hence the Lebesgue dominated convergence theorem implies that the quantity on the right-hand side of (4.14) converges to 0. For the other term, it suffices to note that $C(w(N), x) - B(w, x) = B(x, w) - C(x, w(N)) + \bar{x} \cdot w(N) - \bar{x} \cdot \bar{w}$ and $\lim_{N \rightarrow \infty} E[\|w(N) - w\|_{m,\theta/2}^m] = 0$. \square

Proof of Lemma 4.2. First we prove that for any N ,

$$\varepsilon_N := \inf \left\{ \mu \left(U_N(z^1, \dots, z^l; \varepsilon) \right) \mid \max_{1 \leq i \leq l} \|z^i\|_{m,\theta'/2} \leq r \right\} > 0. \quad (4.15)$$

Note that for any $z^1, \dots, z^l \in W_{m,\theta/2}(\mathbb{R})$,

$$\mu \left(U_N(z^1, \dots, z^l; \varepsilon) \right) > 0. \quad (4.16)$$

If (4.15) does not hold, then we can find a sequence $\{z^{i,n}\}$ such that $\sup_{i,n} \|z^{i,n}\|_{m,\theta'/2} \leq r$ and $\lim_{n \rightarrow \infty} \mu(U_N(z^{1,n}, \dots, z^{l,n}; \varepsilon)) = 0$. Since the embedding $W_{m,\theta'/2}(\mathbb{R}) \subset W_{m,\theta/2}(\mathbb{R})$ is compact, there exists a subsequence $\{z^{i,n(k)}\}$ and $\{y^i\} \subset W_{m,\theta/2}(\mathbb{R})$ such that $\lim_{k \rightarrow \infty} \|z^{i,n(k)} - y^i\|_{m,\theta/2} = 0$. By Lemma 4.3 and $E[\|C(x, w(N))\|_{m,\theta}^m] \leq E[\|B(x, w)\|_{m,\theta}^m]$ and so on,

$$\lim_{k \rightarrow \infty} E \left[\|C(w(N), z^{i,n(k)}) - C(w(N), y^i)\|_{m,\theta} + \|C(z^{i,n(k)}, w(N)) - C(y^i, w(N))\|_{m,\theta} \right] = 0.$$

This implies that $\mu(U_N(y^1, \dots, y^l; \varepsilon/2)) = 0$ which is a contradiction. Next we prove that $\liminf_{N \rightarrow \infty} \varepsilon_N > 0$. The random variable $(w, B(w, z^i), B(z^i, w))$ defines a Gaussian measure with mean 0 on the separable Banach space

$$W_{m,\theta'/2}(\mathbb{R}) \times \prod_{i=1}^{2l} L_{m,\theta}(\Delta \rightarrow \mathbb{R}).$$

Therefore every ball of positive radius has positive measure. See [7]. Thus we obtain for any $\varepsilon > 0$ and $\{z^i\}_{i=1}^l \subset W_{m,\theta/2}(\mathbb{R})$,

$$\mu(U(z^1, \dots, z^l; \varepsilon)) > 0, \quad (4.17)$$

where

$$U(z^1, \dots, z^l; \varepsilon) = \left\{ w \in W_{m,\theta'/2}(\mathbb{R}) \mid \max_{1 \leq i \leq l} \{ \|w\|_{m,\theta'/2}, \|B(w, z^i)\|_{m,\theta}, \|B(z^i, w)\|_{m,\theta} \} < \varepsilon \right\}. \quad (4.18)$$

Now suppose that there exist $\{z^{i,N}\} \subset W_{m,\theta'/2}(\mathbb{R})$ with $\sup_{i,N} \|z^{i,N}\|_{m,\theta'/2} < r$ and

$$\lim_{N \rightarrow \infty} \mu(U_N(z^{1,N}, \dots, z^{l,N}; \varepsilon)) = 0.$$

We may assume that there exists $y^i \in W_{m,\theta/2}(\mathbb{R})$ such that $\lim_{N \rightarrow \infty} \|z^{i,N} - y^i\|_{m,\theta/2} = 0$. We have

$$C(w(N), z^{i,N}) = C(w(N), z^{i,N} - y^i) + B(w, y^i) - (B(w, y^i) - C(w(N), y^i)). \quad (4.19)$$

Also the $\|\cdot\|_{m,\theta}$ norms of $C(w(N), z^{i,N} - y^i)$ and $B(w, y^i) - C(w(N), y^i)$ converge to 0 in probability by Lemma 4.3. This shows $\mu(U(y^1, \dots, y^l; \varepsilon/2)) = 0$ which is a contradiction and we have proved that $\inf_N \varepsilon_N > 0$. \square

The following lemma will be applied to the set $U_k(\xi^{k+1}, \dots, \xi^d, \eta)_{(\xi^1, \dots, \xi^{k-1})}$ which is defined in (4.40).

Lemma 4.4. *Let $d = 1$. That is, we consider the case where $w \in \Omega$ and $\xi \in \Omega_N$ are real-valued functions on $[0, 1]$. Let $0 < \theta < \theta' < 1$ and $m(1 - \theta) > 2$. Let $x \in W_{m,\theta'/2}(\mathbb{R})$, $y^1, \dots, y^{2l} \in W_{m,\theta'/2}(\mathbb{R})$ and $z^1, \dots, z^{2l} \in W_{m,\theta}(\Delta \rightarrow \mathbb{R})$. Let r be a positive number and $0 < \delta < 1$. Suppose that $\|x\|_{m,\theta'/2} < \delta r$ and $\max_{1 \leq i \leq 2l} \|z^i\|_{m,\theta} < \delta r$. Let us consider a bounded open subset of Ω_N ,*

$$\begin{aligned} & U_N(\{y^i\}_{i=1}^{2l}, \{z^i\}_{i=1}^{2l}, x) \\ &= \left\{ \xi \in \Omega_N \mid \|\xi + x\|_{m,\theta'/2} < r, \max_{1 \leq i \leq l} \|C(\xi, y^i) + z^i\|_{m,\theta} < r, \right. \\ & \quad \left. \max_{1 \leq i \leq l} \|C(y^{i+l}, \xi) + z^{i+l}\|_{m,\theta} < r \right\}. \end{aligned} \quad (4.20)$$

(1) *It holds that for any $C > 0$*

$$\inf \left\{ \mu(U_N(\{y^i\}_{i=1}^{2l}, \{z^i\}_{i=1}^{2l}, x)) \mid \max_{1 \leq i \leq 2l} \|y^i\|_{m,\theta'/2} \leq C, N \in \mathbb{N} \right\} > 0. \quad (4.21)$$

(2) *Let $W^1(U_N(\{y^i\}, \{z^i\}, x), \mu_N)$ be the Sobolev space which consists of L^2 -functions with respect to μ_N on $U_N(\{y^i\}, \{z^i\}, x)$ whose weak derivatives are in $L^2(\mu_N)$. This set coincides with $W^1(U_N(\{y^i\}, \{z^i\}, x))$ which is usual Sobolev spaces whose derivatives are in L^2 with respect to the Lebesgue measure. Moreover there exists a bounded linear operator (extension operator) $T : W^1(U_N(\{y^i\}, \{z^i\}, x), \mu_N) \rightarrow W^1(\Omega_N, \mu_N)$ such that $Tf|_{U_N(\{y^i\}, \{z^i\}, x)} = f$.*

(3) It holds that for any $f \in W^1(U_N(\{y^i\}, \{z^i\}, x), \mu_N)$,

$$\text{Var}(f; U_N(\{y^i\}, \{z^i\}, x)) \leq \int_{U_N(\{y^i\}, \{z^i\}, x)} |Df(\xi)|_H^2 d\mu_{N, U_N(\{y^i\}, \{z^i\}, x)}(\xi). \quad (4.22)$$

where $\mu_{N, U_N(\{y^i\}, \{z^i\}, x)}$ is the normalized probability measure of μ_N on $U_N(\{y^i\}, \{z^i\}, x)$.

Proof. Part (1) follows from Lemma 4.2. while (2) follows from the fact that $U_N(\{y^i\}, \{z^i\}, x)$ is a bounded convex domain of Ω_N . Then Part (3) follows from the result in (2) and the Poincaré inequality on a convex domain in a Euclidean space with a Gaussian measure ([13]). \square

From now on, we fix parameters m, θ, θ' as follows.

Assumption 4.5. Let us fix m, θ, θ' such that $m(1 - \theta') > 4$ and $2/3 < \theta < \theta' < 1$.

Let $\varphi = \varphi_t = (\varphi_t^1, \dots, \varphi_t^d)$ ($0 \leq t \leq 1$) be an element of H and define

$$\begin{aligned} U_{r, \varphi} &= \left\{ w \in \Omega \mid \max_{1 \leq i \leq d} \|w^i\|_{m, \theta'/2} < r, \max_{1 \leq j < k \leq d} \|C(w^j, w^k)\|_{m, \theta} < r, \max_{1 \leq i \leq j \leq d} \|C(\varphi^i, w^j)\|_{m, \theta} < r, \right. \\ &\quad \left. \sup_{1 \leq i \leq j \leq d} \|C(w^i, \varphi^j)\|_{m, \theta} < r \right\}, \end{aligned} \quad (4.23)$$

and

$$\begin{aligned} U_r(\varphi) &= \left\{ w \in \Omega \mid \max_{1 \leq i \leq d} \|w^i - \varphi^i\|_{m, \theta'/2} < r, \max_{1 \leq j < k \leq d} \|C(w^j - \varphi^j, w^k - \varphi^k)\|_{m, \theta} < r, \right. \\ &\quad \left. \max_{1 \leq i \leq j \leq d} \|C(\varphi^i, w^j - \varphi^j)\|_{m, \theta} < r, \max_{1 \leq i \leq j \leq d} \|C(w^i - \varphi^i, \varphi^j)\|_{m, \theta} < r \right\}. \end{aligned} \quad (4.24)$$

Although these sets are different from the metric ball in the metric space (Ω, d_Ω) , these play a similar kind of role of the balls in normed linear spaces. Note that we have the following relation:

$$U_r(\varphi) = \{w + \varphi \mid w \in U_{r, \varphi}\}. \quad (4.25)$$

The strict positivity of the measures of these subsets for any $r > 0$ and $\varphi \in H$ can be proved by the argument similar to the proof of Lemma 2.6 in [4]. See [28] also.

Now we state our Poincaré's lemmas.

Theorem 4.6. Let $\beta \in \mathbb{D}^{\infty, q}(W^d, H^*) \cap L^2(W^d, H^*)$, where $q > 1$. Suppose that $d\beta = 0$ on $U_{r, \varphi}$. Then for any $r' < r$, there exists $g \in \mathbb{D}^{\infty, q}(W^d, \mathbb{R}) \cap \mathbb{D}^{1, 2}(W^d, \mathbb{R})$ such that $dg = \beta$ on $U_{r', \varphi}$.

Theorem 4.7. Let $\beta \in \mathbb{D}^{\infty, q}(W^d, H^*) \cap L^2(W^d, H^*)$, where $q > 1$. We assume that the first derivative of φ is a bounded variation function. Suppose that $d\beta = 0$ on $U_r(\varphi)$. Then for any $r' < r$, there exists $g \in \mathbb{D}^{\infty, q}(W^d, \mathbb{R}) \cap \mathbb{D}^{1, 2}(W^d, \mathbb{R})$ such that $dg = \beta$ on $U_{r'}(\varphi)$.

First we prove Theorem 4.7 using Theorem 4.6. After that, we will prove Theorem 4.6.

Proof of Theorem 4.7. Let $T_\varphi w = w + \varphi$. Then $U_r(\varphi) = \{T_\varphi w \mid w \in U_{r,\varphi}\}$. For a measurable function u on W^d , define $T_\varphi^* u(w) = u(w + \varphi)$. Let χ_R be a smooth function on \mathbb{R} such that $\chi_R(x) = 1$ for $|x| \leq R$ and $\chi_R(x) = 0$ and $|x| \geq R + 1$. Let $\hat{\chi}_R(w) = \chi(\|w\|_{m,\theta'/2}^m)$. Note that $D^l \hat{\chi}_R(w)$ is a bounded function for all l . This follows from Lemma 3.8. For any $q > 1, k \in \mathbb{N} \cup \{0\}$, there exist positive constants C_1, C_2 ($C_1 < C_2$) such that for any $u \in \mathbb{D}^{k,q}(W^d)$

$$C_1 \|u\|_{k,q} \leq \|(T_\varphi^* u) \hat{\chi}_R\|_{k,q} \leq C_2 \|u\|_{k,q}.$$

This can be checked by using the Cameron-Martin formula and the fact that the stochastic integral $\int_0^1 (\varphi'(t), dw(t))$ is actually a Riemann-Stieltjes integral and bounded on $\{w \in \Omega \mid \|w\|_{m,\theta/2} \leq R + 1\}$. The same estimates hold for 1-forms. Let β be the 1-form which satisfies the assumptions of the theorem. Let R be a sufficiently large number and set $\bar{\beta} = (T_\varphi^* \beta) \hat{\chi}_R$. Then $\bar{\beta} \in \mathbb{D}^{\infty,q}(W^d, H^*) \cap L^2(W^d, H^*)$ and $d\bar{\beta} = 0$ on $U_{r,\varphi}$. Therefore by Theorem 4.6, there exists $\bar{g} \in \mathbb{D}^{\infty,q}(W^d, H^*) \cap \mathbb{D}^{1,2}(W^d, H^*)$ such that $d\bar{g} = \bar{\beta}$ on $U_{r',\varphi}$. Define $g = (T_{-\varphi}^* \bar{g}) \hat{\chi}_{R'}$, where R' is also a sufficiently large positive number. Then g satisfies the desired properties. \square

To prove Theorem 4.6, we need some homotopy arguments on finite dimensional space. Let U be a bounded open subset of \mathbb{R}^{n+m} . Let us write $z = (x, y) \in \mathbb{R}^{n+m}$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Let A be the image of the projection of U with respect to the first variable x . Clearly, A is also an open subset. For $x \in A$, set $U_x = \{y \in \mathbb{R}^m \mid (x, y) \in U\}$ which is also an open subset. Using the notation above, we prepare the following. The proof of this result is easy and we omit it.

Lemma 4.8. *Suppose that U_x is a convex set and contains 0. Let α be a C^∞ 1-form on U . We write*

$$\alpha(z) = \sum_{i=1}^n \beta_i(x, y) dx^i + \sum_{j=1}^m \gamma_j(x, y) dy^j. \quad (4.26)$$

Let $\pi : U \rightarrow A$ be the projection and define $s : A \rightarrow U$ by $s(x) = (x, 0) \in U$ for $x \in A$. Let

$$(K\alpha)(z) = \int_0^1 \sum_{j=1}^m \gamma_j(x, ty) y^j dt. \quad (4.27)$$

If $d\alpha = 0$ on U , then it holds that $s^ \alpha$ is a closed form on A and*

$$\alpha = \pi^* s^* \alpha + dK\alpha. \quad (4.28)$$

Needless to say, if $H^1(A, \mathbb{R}) = 0$, then there exists a smooth function g on A such that $dg = s^* \alpha$. Therefore we have $\alpha = d(\pi^* g + K\alpha)$. We use this in the proof of Theorem 4.6.

Proof of Theorem 4.6. Let $N \in \mathbb{N}$ and set

$$\begin{aligned} R_N = & \left\{ \eta = (\eta^1, \dots, \eta^d) \in \Omega_N^\perp \mid \max_{1 \leq i \leq d} \|\eta^i\|_{m,\theta'/2} < r/4, \right. \\ & \max_{1 \leq i < j \leq d} \|C(\eta^i, \eta^j)\|_{m,\theta} < r/4, \max_{1 \leq i \leq j \leq d} \|C(\varphi^i, \eta^j)\|_{m,\theta} < r/4, \\ & \left. \max_{1 \leq i \leq j \leq d} \|C(\eta^i, \varphi^j)\|_{m,\theta} < r/4 \right\}. \end{aligned} \quad (4.29)$$

For $\eta \in \Omega_N^\perp$, define

$$U_{r,\varphi}(\eta) = \left\{ \xi = (\xi^1, \dots, \xi^d) \in \Omega_N \mid \xi + \eta \in U_{r,\varphi}, \max_{1 \leq i < j \leq d} \|C(\xi^i, \eta^j)\|_{m,\theta} < r/4, \right. \\ \left. \max_{1 \leq i < j \leq d} \|C(\eta^i, \xi^j)\|_{m,\theta} < r/4 \right\}. \quad (4.30)$$

This set can be identified with a bounded open subset of the Euclidean space of dimension $2^N d$. Using this, we define an approximate set of $U_{r,\varphi}$ as follows.

$$U_{r,\varphi,N} = \left\{ w \in \Omega \mid w(N) \in U_{r,\varphi}(w(N)^\perp), w(N)^\perp \in R_N \right\}. \quad (4.31)$$

Since Ω is isomorphic to the product space $\Omega_N \times \Omega_N^\perp$, $U_{r,\varphi,N}$ is thought as a subset of this product space. Thus any function g on $U_{r,\varphi,N}$ can be identified with a function of (ξ, η) where $\xi \in U_{r,\varphi}(\eta), \eta \in R_N$.

Using Lemma 4.1 and Lemma 4.2 and an induction, we prove the following Claims.

Claim 1 Let $\eta \in R_N$. Poincaré's inequality holds on $U_{r,\varphi}(\eta)$ in the following form:

$$\text{Var}(g; U_{r,\varphi}(\eta)) \leq C \int_{U_{r,\varphi}(\eta)} |Dg(\xi)|_H^2 d\mu_{N,U_{r,\varphi}(\eta)}(\xi), \quad (4.32)$$

where C is a positive constant which depends only on $r, d, \varphi, m, \theta, \theta'$ and $\mu_{N,U_{r,\varphi}(\eta)}$ is a normalized probability measure on $U_{r,\varphi}(\eta)$.

Claim 2 There exists a measurable function g_N on $U_{r,\varphi,N}$ such that for μ_N^\perp -almost all $\eta \in R_N$, the function $\xi \in U_{r,\varphi}(\eta) \rightarrow g_N(\xi, \eta)$ is a C^∞ function with

$$\sup_{\xi \in U_{r,\varphi}(\eta)} |g_N(\xi, \eta)| < \infty \quad (4.33)$$

$$\int_{U_{r,\varphi}(\eta)} g_N(\xi, \eta) d\mu_N(\xi) = 0 \quad (4.34)$$

and $d_N g_N = \beta_N$ holds on $U_{r,\varphi,N}$. Here $d_N g_N$ is the exterior differential of g_N with respect to the variable ξ and $\beta_N = P_N \beta$ which is the projection of β onto $(\Omega_N \cap H)^*$.

To prove these claims, we introduce the following sets. First, we fix $\eta \in R_N$. Let

$$B_{d,N}(\eta) = \left\{ \xi^d \mid \|\xi^d + \eta^d\|_{m,\theta'/2} < r, \max_{1 \leq i \leq d} \|C(\varphi^i, \xi^d + \eta^d)\|_{m,\theta} < r, \right. \\ \left. \|C(\xi^d + \eta^d, \varphi^d)\|_{m,\theta} < r, \max_{1 \leq l < d} \|C(\eta^l, \xi^d)\|_{m,\theta} < r/4 \right\}. \quad (4.35)$$

For $1 \leq k \leq d-1$, taking $\xi^i \in B_{i,N}(\xi^{i+1}, \dots, \xi^d, \eta)$ ($k+1 \leq i \leq d$) inductively, we define

$$B_{k,N}(\xi^{k+1}, \dots, \xi^d, \eta) \\ = \left\{ \xi^k \mid \|\xi^k + \eta^k\|_{m,\theta'/2} < r, \max_{l > k} \|C(\xi^k + \eta^k, \xi^l + \eta^l)\|_{m,\theta} < r, \right. \\ \max_{1 \leq i \leq k} \|C(\varphi^i, \xi^k + \eta^k)\|_{m,\theta} < r, \max_{l \geq k} \|C(\xi^k + \eta^k, \varphi^l)\|_{m,\theta} < r, \\ \max_{l > k} \|C(\xi^k, \eta^l)\|_{m,\theta} < r/4, \max_{1 \leq j < k} \|C(\eta^j, \xi^k)\|_{m,\theta} < r/4 \left. \right\}. \quad (4.36)$$

Note that $0 \in B_{k,N}(\xi^{k+1}, \dots, \xi^d, \eta)$. We denote all elements $(\xi^{k+1}, \dots, \xi^d)$ which can be obtained in this way by $S_{k+1,d}(\eta)$.

Now we define a sequence of subsets inductively. First set $U_d(\eta) = U_{r,\varphi}(\eta)$. Inductively, for $1 \leq k \leq d-1$ and $(\xi^{k+1}, \dots, \xi^d) \in S_{k+1,d}(\eta)$ define

$$\begin{aligned} & U_k(\xi^{k+1}, \dots, \xi^d, \eta) \\ &= \left\{ (\xi^1, \dots, \xi^k) \mid \begin{aligned} & \max_{1 \leq i \leq k} \|\xi^i + \eta^i\|_{m,\theta'/2} < r, \\ & \max_{1 \leq i < j \leq k} \|C(\xi^i + \eta^i, \xi^j + \eta^j)\|_{m,\theta} < r, \max_{1 \leq i \leq k < l \leq d} \|C(\xi^i + \eta^i, \xi^l + \eta^l)\|_{m,\theta} < r, \\ & \max_{1 \leq i \leq j \leq k} \|C(\varphi^i, \xi^j + \eta^j)\|_{m,\theta} < r, \max_{1 \leq i \leq k, i \leq j} \|C(\xi^i + \eta^i, \varphi^j)\|_{m,\theta} < r, \\ & \max_{1 \leq i \leq k, i < j \leq d} \|C(\xi^i, \eta^j)\|_{m,\theta} < r/4, \max_{1 \leq i < j, 1 < j \leq k} \|C(\eta^i, \xi^j)\|_{m,\theta} < r/4 \end{aligned} \right\}. \end{aligned} \quad (4.37)$$

Then

$$B_{k,N}(\xi^{k+1}, \dots, \xi^d, \eta) = \left\{ \xi^k \mid U_k(\xi^{k+1}, \dots, \xi^d, \eta)^{\xi^k} \neq \emptyset \right\} \quad (4.38)$$

and for $\xi^k \in B_{k,N}(\xi^{k+1}, \dots, \xi^d, \eta)$,

$$U_k(\xi^{k+1}, \dots, \xi^d, \eta)^{\xi^k} = U_{k-1}(\xi^k, \dots, \xi^d, \eta). \quad (4.39)$$

In the above and below, $U_k(\dots)^{\xi^k}$, $U_k(\dots)_{(\xi^1, \dots, \xi^{k-1})}$ denote the sections as in Lemma 4.1. Also

$$\begin{aligned} & U_{k-1}(0, \xi^{k+1}, \dots, \xi^d, \eta) \\ &= \left\{ (\xi^1, \dots, \xi^{k-1}) \mid U_k(\xi^{k+1}, \dots, \xi^d, \eta)_{(\xi^1, \dots, \xi^{k-1})} \neq \emptyset \right\} \end{aligned}$$

and for $(\xi^1, \dots, \xi^{k-1}) \in U_{k-1}(0, \xi^{k+1}, \dots, \xi^d, \eta)$,

$$\begin{aligned} & U_k(\xi^{k+1}, \dots, \xi^d, \eta)_{(\xi^1, \dots, \xi^{k-1})} \\ &= \left\{ \xi^k \mid \begin{aligned} & \|\xi^k + \eta^k\|_{m,\theta'/2} < r, \max_{1 \leq i < k} \|C(\xi^i + \eta^i, \xi^k + \eta^k)\|_{m,\theta} < r, \\ & \max_{l > k} \|C(\xi^k + \eta^k, \xi^l + \eta^l)\|_{m,\theta} < r \\ & \max_{1 \leq i \leq k} \|C(\varphi^i, \xi^k + \eta^k)\|_{m,\theta} < r, \max_{l \geq k} \|C(\xi^k + \eta^k, \varphi^l)\|_{m,\theta} < r, \\ & \max_{l > k} \|C(\xi^k, \eta^l)\|_{m,\theta} < r/4, \max_{1 \leq i < k} \|C(\eta^i, \xi^k)\|_{m,\theta} < r/4 \end{aligned} \right\}. \end{aligned} \quad (4.40)$$

Note that $U_k(\xi^{k+1}, \dots, \xi^d, \eta)_{(\xi^1, \dots, \xi^{k-1})}$ is a convex set of \mathbb{R}^{2^N} and contains 0. Further, by Lemma 4.2, we have for all $1 \leq k \leq d-1$,

$$\begin{aligned} & \inf \left\{ \mu \left(U_k(\xi^{k+1}, \dots, \xi^d, \eta)_x \cap U_k(\xi^{k+1}, \dots, \xi^d, \eta)_y \right) \mid x, y \in U_{k-1}(0, \xi^{k+1}, \dots, \xi^d, \eta), \right. \\ & \quad \left. (\xi^{k+1}, \dots, \xi^d) \in S_{k+1,d}(\eta), \eta \in R_N \right\} > 0 \end{aligned} \quad (4.41)$$

and the lower bound is given by the inverse of products of $C(l, r/4, r, m, \theta, \theta')$. Hence in order to check Claim 1, by (4.39) and Lemma 4.1, we need to prove Poincaré's inequality with a Poincaré constant which is independent of $\xi^k, \dots, \xi^d, \eta$ on $U_{k-1}(\xi^k, \dots, \xi^d, \eta)$. This is checked by using Lemma 4.4. Thus we see that Claim 1 holds with the constant C which depends only on the inverse of products of $C(l, r/4, r, m, \theta, \theta')$.

We prove Claim 2. Let $\eta \in R_N$. Then $\beta_N(\cdot, \eta) \in \wedge^1 T^* U_{r, \varphi}(\eta)$ is also a closed C^∞ -differential form and the supremum norm of all derivatives are finite for almost all η by the Sobolev embedding theorem. By Lemma 4.8 and using inductive argument, we can construct a bounded function $u_N(\cdot, \eta) \in C^\infty(U_{r, \varphi}(\eta))$ explicitly such that $d_N u_N = \beta_N$ and $u_N(\xi, \eta)$ is a measurable function on $U_{r, \varphi, N}$. Using u_N , we see that

$$g_N = u_N - \frac{1}{\mu_N(U_{r, \varphi}(\eta))} \int_{U_{r, \varphi}(\eta)} u_N(\xi, \eta) d\mu_N(\xi)$$

is the desired function.

Now, we prove the existence of g which satisfies the desired property in the Theorem. Let g_N be the function in the Claim 2. Then by the Poincaré inequality established in the Claim 1, it holds that

$$\|g_N\|_{L^2(U_{r, \varphi, N})}^2 \leq C \|\beta_N\|_{L^2(U_{r, \varphi, N})}^2 \leq C \|\beta\|_{L^2(U_{r, \varphi})}^2. \quad (4.42)$$

Let $\hat{g}_N(w) = g_N(w) 1_{U_{r, \varphi, N}}(w)$. Let us choose a positive numbers r_1, r_2 such that $0 < r' < r_1 < r_2 < r$. Let ρ be a smooth function on $\mathbb{R}^{3d(d+1)/2}$ such that $\max_y |\rho(y) - \max_i |y^i||$ is sufficiently small. It is easy to see the existence of such a function using a mollifier. Then there exists a small positive number ε such that for any $r_1 \leq s \leq r_2$,

$$\begin{aligned} \left\{ x = (x^i) \in \mathbb{R}^{3d(d+1)/2} \mid \max_i |x^i| < r' + \varepsilon \right\} &\subset \left\{ x = (x^i) \in \mathbb{R}^{3d(d+1)/2} \mid \rho(x^{(m)}) < s^m \right\} \\ &\subset \left\{ x = (x^i) \in \mathbb{R}^{3d(d+1)/2} \mid \max_i |x^i| < r \right\}, \end{aligned} \quad (4.43)$$

where $x^{(m)} = ((x^1)^m, \dots, (x^{3d(d+1)/2})^m)$. Note that the index j of $(x^i)^j$ is the power and i stands for the i -th element. Let $\hat{\rho}(w)$ be the composition of ρ and the $3d(d+1)/2$ random variables

$$\begin{aligned} \|w^i\|_{m, \theta'/2}^m \ (1 \leq i \leq d), \|C(w^j, w^k)\|_{m, \theta}^m \ (1 \leq j < k \leq d) \\ \|C(\varphi^i, w^j)\|_{m, \theta}^m \ (1 \leq i \leq j \leq d), \|C(w^i, \varphi^j)\|_{m, \theta}^m \ (1 \leq i \leq j \leq d). \end{aligned} \quad (4.44)$$

Let χ be the smooth decreasing function such that $\chi(u) = 1$ for $u \leq (r/6)^m$ $\chi(u) = 0$ for $u \geq (r/5)^m$ and set

$$\begin{aligned} \hat{\chi}_N(w) &= \chi \left(\sum_{i=1}^d \|w(N)^{\perp, i}\|_{m, \theta'/2}^m + \sum_{1 \leq j < k \leq d} \|C(w(N)^{\perp, j}, w(N)^{\perp, k})\|_{m, \theta}^m \right. \\ &\quad + \sum_{1 \leq i \leq j \leq d} \|C(\varphi^i, w(N)^{\perp, j})\|_{m, \theta}^m + \sum_{1 \leq i \leq j \leq d} \|C(w(N)^{\perp, i}, \varphi^j)\|_{m, \theta}^m \\ &\quad \left. + \sum_{1 \leq i < j \leq d} \|C(w(N)^i, w(N)^{\perp, j})\|_{m, \theta}^m + \sum_{1 \leq i < j \leq d} \|C(w(N)^{\perp, i}, w(N)^j)\|_{m, \theta}^m \right). \end{aligned}$$

Let ψ be the smooth decreasing function such that $\psi(u) = 1$ for $u \leq \frac{r_1^m + r_2^m}{2}$ and $\psi(u) = 0$ for $u \geq \frac{r_1^m + 2r_2^m}{3}$. Let $h_N(w) = \hat{g}_N(w)\psi(\hat{\rho}(w))\hat{\chi}_N(w)$. Since $\sup_N \|\hat{g}_N\|_{L^2(W^d, \mu)} < \infty$, there exists a subsequence $\hat{g}_{N(k)}$ ($N(1) < N(2) < \dots$) such that $\hat{g}_{N(k)}$ converges weakly to some $\hat{g}_\infty \in L^2(W^d, \mu)$. Noting that $\|\hat{\chi}_N\|_\infty \leq 1$ and $\lim_{N \rightarrow \infty} \hat{\chi}_N(w) = 1$ for all $w \in \Omega$, we see that $\hat{g}_{N(k)}(w)\psi(\hat{\rho}(w))\hat{\chi}_{N(k)}(w)$ also converges weakly to $\hat{g}_\infty(w)\psi(\hat{\rho}(w))$ which we denote by $h_\infty(w)$. We calculate the weak derivative of h_∞ . Fix a natural number N_0 and let $\theta \in \mathbb{D}^\infty(W^d \rightarrow P_{N_0}H^*)$. Then

$$\begin{aligned} \int_{W^d} h_\infty(w) D^* \theta(w) d\mu(w) &= \lim_{k \rightarrow \infty} \int_{W^d} h_{N(k)}(w) D^* \theta(w) d\mu(w) \\ &= \lim_{k \rightarrow \infty} \int_{W^d} (d_{N(k)} h_{N(k)}(w), \theta(w)) d\mu(w). \end{aligned} \quad (4.45)$$

Here

$$\begin{aligned} d_{N(k)} (\hat{g}_{N(k)} \psi(\hat{\rho}) \hat{\chi}_{N(k)}) &= \beta_{N(k)} \psi(\hat{\rho}) \hat{\chi}_{N(k)} + \hat{g}_{N(k)} d_{N(k)} (\psi(\hat{\rho}(w))) \hat{\chi}_{N(k)}(w) \\ &\quad + \hat{g}_{N(k)} \psi(\hat{\rho}(w)) d_{N(k)} \hat{\chi}_{N(k)}(w). \end{aligned} \quad (4.46)$$

Noting that

$$\lim_{k \rightarrow \infty} \|d_{N(k)} (\psi(\hat{\rho})) - d(\psi(\hat{\rho}))\|_{L^4(\mu)} = 0, \quad (4.47)$$

$$\lim_{k \rightarrow \infty} \|d_{N(k)} \hat{\chi}_{N(k)}\|_{L^4(\mu)} = 0, \quad (4.48)$$

we get

$$\begin{aligned} \int_{W^d} h_\infty(w) D^* \theta(w) d\mu(w) &= \int_{W^d} (\beta(w) \psi(\hat{\rho}(w)) + \hat{g}_\infty(w) d(\psi(\hat{\rho}(w))), \theta(w)) d\mu(w). \end{aligned} \quad (4.49)$$

This implies $dh_\infty = \beta\psi(\hat{\rho}) + \hat{g}_\infty d(\psi(\hat{\rho}))$ in weak sense. By Lemma 3.4 and Lemma 3.8, $d(\psi(\hat{\rho}))$ is a bounded function. Hence $dh_\infty \in L^2(W^d, \mu)$ which implies $h_\infty \in \mathbb{D}^{1,2}(W^d, \mathbb{R})$. Also h_∞ satisfies that $dh_\infty = \beta$ on $U_{r', \varphi}$. Finally we need to show the regularity of the higher order derivatives of h_∞ . Choosing a smooth function ψ_1 on \mathbb{R} such that $\psi_1(u) = 1$ for $u \leq \frac{r_1^m + 3r_2^m}{4}$ and $\psi_1(u) = 0$ for $u \geq \frac{r_1^m + 4r_2^m}{5}$, we have

$$\hat{g}_\infty \psi_1(\hat{\rho}) d(\psi(\hat{\rho})) = \hat{g}_\infty d(\psi(\hat{\rho})).$$

We see that $\hat{g}_\infty \psi_1(\hat{\rho}) \in \mathbb{D}^{1,2}(W^d, \mathbb{R})$ by the same argument as the above. Hence $h_\infty \in \mathbb{D}^{2,q}(W^d, \mathbb{R})$. Iterating this procedure, we get $h_\infty \in \mathbb{D}^{\infty,q}(W^d, \mathbb{R})$. \square

Remark 4.9. In the same way as the proof of Claim 1, we can prove that for any $g \in \mathbb{D}^{1,2}(W^d)$,

$$\text{Var}(g; U_{r, \varphi}) \leq C \int_{U_{r, \varphi}} |Dg(w)|_H^2 d\mu_{U_{r, \varphi}}(w), \quad (4.50)$$

where $\mu_{U_{r, \varphi}}$ denotes the normalized probability measure on $U_{r, \varphi}$ and Var denotes the variance with respect to the measure. We may define a local Sobolev space $W^1(U_{r, \varphi})$. It is not clear that $W^1(U_{r, \varphi})$ coincides with the restriction of $\mathbb{D}^{1,2}(W^d)$ to $U_{r, \varphi}$ at the moment. Note that the extension property of functions on convex sets were studied in [20]. See [21] for more recent results.

Let $B_\varepsilon(e) = \{a \in G \mid d(a, e) < \varepsilon\}$. We assume that ε is sufficiently small and $B_\varepsilon(e)$ is diffeomorphic to a standard ball in a Euclidean space. Let

$$\mathcal{D}_\varepsilon = \{w \in \Omega \mid X(1, e, w) \in B_\varepsilon(e)\}.$$

This set is formally homotopy equivalent to $S = \{w \in \Omega \mid X(1, e, w) = e\}$ and $L_e(G)$. We construct a covering of \mathcal{D}_ε by a countable family of $U_r(\varphi)$ in the next section. This covering is vital for the proof of the existence of f satisfying $df = \alpha$.

5 A covering lemma for \mathcal{D}_ε

For $K \in \mathbb{N}$ and $0 < \kappa < 1$, let

$$\begin{aligned} A_K &= \{w \in \Omega \mid d_\Omega(0, w) < K\} \\ B_{N, \kappa} &= \left\{w \in \Omega \mid \max_i \|w(N)^{\perp, i}\|_{m, \theta'/2} < \kappa, \max_{1 \leq i < j \leq d} \|C(w(N)^{\perp, i}, w(N)^{\perp, j})\|_{m, \theta} < \kappa, \right. \\ &\quad \left. \max_{1 \leq i \leq j \leq d} \|C(w(N)^i, w(N)^{\perp, j})\|_{m, \theta} < \kappa, \max_{1 \leq i \leq j \leq d} \|C(w(N)^{\perp, i}, w(N)^j)\|_{m, \theta} < \kappa\right\}. \end{aligned} \quad (5.1)$$

Note that $A_K = U_K(0)$, $B_{N, \kappa} = \{w \in \Omega \mid w \in U_\kappa(w(N))\}$. For $w \in A_K \cap B_{N, \kappa}$, $\max_i \|w(N)^i\|_{m, \theta'/2} < K + 1$. Let $\varepsilon_n = \varepsilon(1 - \frac{1}{n})$ ($n = 1, 2, \dots$) and

$$\mathcal{D}_{\varepsilon_n, K, N, \kappa} = \mathcal{D}_{\varepsilon_n} \cap A_K \cap B_{N, \kappa} \quad (5.3)$$

$$(5.4)$$

For any $\kappa > 0, n, K$, we have

$$\liminf_{N \rightarrow \infty} \mathcal{D}_{\varepsilon_n, K, N, \kappa} = \mathcal{D}_{\varepsilon_n} \cap A_K. \quad (5.5)$$

For fixed n and K , we can find a positive number $\kappa(n, K)$ such that there exists a finite cover of $\mathcal{D}_{\varepsilon_n, K, N, \kappa(n, K)}$ by $U_r(\varphi)$ which satisfies $U_r(\varphi) \subset \mathcal{D}_{\varepsilon_{2n}}$. Since (5.5) holds, this implies that there exists a countable cover of $\mathcal{D}_{\varepsilon_n} \cap A_K$ by $U_r(\varphi)$ which are included in $\mathcal{D}_{\varepsilon_{2n}}$ and so does for \mathcal{D}_ε too. More precisely we prove the following.

Lemma 5.1. (1) Let $R_{m, \theta} = \max(M_{m, \theta}^2, N_{m, \theta})$. See Lemma 3.4(1) and Lemma 3.5 for the constants $M_{m, \theta}, N_{m, \theta}$. Let

$$\kappa < \min \left(\frac{\varepsilon}{48nR_{m, \theta}(K+1)F(K+18R_{m, \theta}(K+1))}, \frac{1}{2} \right), \quad (5.6)$$

where F is a function which appeared in Proposition 3.7. Let $w \in \mathcal{D}_{\varepsilon_n, K, N, \kappa}$. We take $\varphi \in H$ such that

$$\|\varphi - w(N)\|_H \leq \frac{\kappa}{3(6\kappa + 2K + 5)}. \quad (5.7)$$

Then

$$w \in U_{4\kappa/3}(\varphi) \subset U_{\sqrt{2}\kappa}(\varphi) \subset \mathcal{D}_{\varepsilon_{2n}}. \quad (5.8)$$

(2) Let κ be a positive number satisfying (5.6). Then for any $N \in \mathbb{N}$, there exists $L = L(n, K, N, \kappa)$ and a finite number of piecewise linear paths $\{\varphi_i\}_{i=1}^L \subset \Omega_N$ such that

$$\mathcal{D}_{\varepsilon_n, K, N, \kappa} \subset \cup_{i=1}^L U_{4\kappa/3}(\varphi_i) \subset \cup_{i=1}^L U_{\sqrt{2}\kappa}(\varphi_i) \subset \mathcal{D}_{\varepsilon_{2n}}. \quad (5.9)$$

(3) Let $\{\kappa_i, \varphi_i\}_{i=1}^\infty$ be countable positive numbers and piecewise linear paths which are obtained in (2) when N, K, n take all values of natural numbers. Then it holds that

$$\mathcal{D}_\varepsilon = \cup_{i=1}^\infty U_{4\kappa_i/3}(\varphi_i) = \cup_{i=1}^\infty U_{\sqrt{2}\kappa_i}(\varphi_i). \quad (5.10)$$

We need a lemma to prove the above.

For $z \in \Omega$, let us define

$$V_r(z) = \{w \in \Omega \mid d_\Omega(w, z) < r\}. \quad (5.11)$$

Lemma 5.2. Let $r > 0$.

(1) Let $\varphi_1 = (\varphi_1^1, \dots, \varphi_1^d), \varphi_2 = (\varphi_2^1, \dots, \varphi_2^d) \in H$. Let $0 < \delta < 1$. If

$$\max_i \|\varphi_1^i - \varphi_2^i\|_H \leq \frac{\delta r}{1 + 3r + 2 \max_i \|\varphi_1^i\|_{m, \theta/2}} \quad (5.12)$$

then $U_r(\varphi_1) \subset U_{(1+\delta)r}(\varphi_2)$.

If the stronger assumption

$$\max_i \|\varphi_1^i - \varphi_2^i\|_H \leq \frac{\delta r}{1 + 6r + 2 \max_i (\|\varphi_1^i\|_{m, \theta/2}, \|\varphi_2^i\|_{m, \theta/2})} \quad (5.13)$$

holds, then we have

$$U_r(\varphi_1) \subset U_{(1+\delta)r}(\varphi_2) \subset U_{(1+\delta)^2 r}(\varphi_1).$$

(2) Let $0 < r < 1$ and $\varphi \in H$. Then

$$U_r(\varphi) \subset V_{R_{m, \theta}(5+6\|\varphi\|_{m, \theta/2})r}(\varphi). \quad (5.14)$$

Proof. (1) Let $\varepsilon = \max_i \|\varphi_1^i - \varphi_2^i\|_H$. Let $w \in U_r(\varphi_1)$. Then we have

$$\|w^i - \varphi_2^i\|_{m, \theta'/2} \leq \|w^i - \varphi_1^i\|_{m, \theta'/2} + \|\varphi_1^i - \varphi_2^i\|_{m, \theta'/2} < r + \varepsilon, \quad (5.15)$$

$$\begin{aligned} & \|C(w^j - \varphi_2^j, w^k - \varphi_2^k)\|_{m, \theta} \\ &= \left\| C(w^j - \varphi_1^j, w^k - \varphi_1^k) + C(\varphi_1^j - \varphi_2^j, w^k - \varphi_1^k) + C(w^j - \varphi_1^j, \varphi_1^k - \varphi_2^k) \right. \\ & \quad \left. + C(\varphi_1^j - \varphi_2^j, \varphi_1^k - \varphi_2^k) \right\|_{m, \theta} \\ &< r + 3\varepsilon r + \varepsilon^2, \end{aligned} \quad (5.16)$$

$$\begin{aligned} \|C(\varphi_2^i, w^j - \varphi_2^j)\|_{m, \theta} &= \left\| C(\varphi_1^i, w^j - \varphi_1^j) + C(\varphi_1^i, \varphi_1^j - \varphi_2^j) + C(\varphi_2^i - \varphi_1^i, w^j - \varphi_1^j) \right. \\ & \quad \left. + C(\varphi_2^i - \varphi_1^i, \varphi_1^j - \varphi_2^j) \right\|_{m, \theta} \\ &< r + \varepsilon \|\varphi_1^i\|_{m, \theta/2} + 2\varepsilon r + \varepsilon^2. \end{aligned} \quad (5.17)$$

In the above, we have used Lemma 3.4 (2). Similarly,

$$\|C(w^i - \varphi_2^i, \varphi_2^j)\|_{m,\theta} < r + \varepsilon r + 2\varepsilon\|\varphi_1^j\|_{m,\theta/2} + \varepsilon^2. \quad (5.18)$$

Therefore if

$$\varepsilon \left(3r + 1 + 2 \max_i \|\varphi_1^i\|_{m,\theta/2} \right) \leq \delta r,$$

then $w \in U_{r(1+\delta)}(\varphi_2)$ which proves the first statement. The second statement follows from the first one.

(2) Assume $w \in U_r(\varphi)$. Let $i < j$. Since $C(w^i, w^j) - C(\varphi^i, \varphi^j) = C(w^i - \varphi^i, w^j - \varphi^j) + C(\varphi^i, w^j - \varphi^j) + C(w^i - \varphi^i, \varphi^j)$, noting Lemma 3.5, we have

$$\|C(w^i, w^j) - C(\varphi^i, \varphi^j)\|_{H,\theta} < 4N_{m,\theta}r(1 + \|\varphi^i\|_{m,\theta/2}).$$

Note that $C(w^i - \varphi^i, w^j - \varphi^j)$ is a limit of iterated integrals of smooth paths and so we can still apply Lemma 3.5. Let us consider the case where $i = j$. Since

$$\begin{aligned} & C(w^i, w^i)_{s,t} - C(\varphi^i, \varphi^i)_{s,t} \\ &= \frac{1}{2} \{ (w^i - \varphi^i)_t - (w^i - \varphi^i)_s \}^2 + C(\varphi^i, w^i - \varphi^i)_{s,t} + C(w^i - \varphi^i, \varphi^i)_{s,t}, \end{aligned} \quad (5.19)$$

$$\begin{aligned} & \|C(w^i, w^i) - C(\varphi^i, \varphi^i)\|_{H,\theta} \\ & \leq \frac{1}{2} \|w^i - \varphi^i\|_{H,\theta/2}^2 + \|C(\varphi^i, w^i - \varphi^i)\|_{H,\theta} + \|C(w^i - \varphi^i, \varphi^i)\|_{H,\theta} \\ & \leq \frac{1}{2} M_{m,\theta}^2 r^2 + 2N_{m,\theta}(1 + \|\varphi^i\|_{m,\theta/2})r. \end{aligned} \quad (5.20)$$

Let $i > j$. Using (3.4), we have

$$\begin{aligned} & C(w^i, w^j)_{s,t} - C(\varphi^i, \varphi^j)_{s,t} \\ &= C(\varphi^j, \varphi^i)_{s,t} - C(w^j, w^i)_{s,t} + \{ (w^i - \varphi^i)_t - (w^i - \varphi^i)_s \} \{ (w^j - \varphi^j)_t - (w^j - \varphi^j)_s \} \\ &+ (\varphi_t^i - \varphi_s^i) \{ (w^j - \varphi^j)_t - (w^j - \varphi^j)_s \} + \{ (w^i - \varphi^i)_t - (w^i - \varphi^i)_s \} (\varphi_t^j - \varphi_s^j). \end{aligned} \quad (5.21)$$

Hence

$$\|C(w^i, w^j) - C(\varphi^i, \varphi^j)\|_{m,\theta} \leq 4N_{m,\theta}r(1 + \|\varphi^i\|_{m,\theta/2}) + M_{m,\theta}^2 r^2 + 2rM_{m,\theta}^2 \max_i \|\varphi^i\|_{m,\theta/2}$$

which completes the proof of (5.14). \square

Proof of Lemma 5.1. (1) Suppose that $w \in \mathcal{D}_{\varepsilon_n, K, N, \kappa}$. Then $\|w(N)\|_{m,\theta'/2} < K + 1$. By Lemma 5.2 (2), $d_\Omega(w(N), w) < 6R_{m,\theta}(K + 1)\kappa$. Hence $d_\Omega(w(N), 0) \leq K + 6R_{m,\theta}(K + 1)\kappa$. By Proposition 3.7 (2),

$$\begin{aligned} d(X(1, e, w(N)), e) & \leq d(X(1, e, w(N)), X(1, e, w)) + d(X(1, e, w), e) \\ & < 6R_{m,\theta}(K + 1)\kappa F(K + 6R_{m,\theta}(K + 1)\kappa) + \varepsilon_n. \end{aligned} \quad (5.22)$$

Hence, if

$$\kappa < \kappa(n, p, K, \varepsilon) := \min \left(\frac{\varepsilon}{6npR_{m,\theta}(K + 1)F(K + 6R_{m,\theta}(K + 1)\kappa)}, 1 \right),$$

then $X(1, e, w(N)) \in B_{\varepsilon(1-\frac{1}{n}(1-\frac{1}{p}))}(e)$. Now assume that $\kappa < 1/2$. Let $z \in U_{2\kappa}(w(N))$. Then $d_{\Omega}(w(N), z) < 12R_{m,\theta}(K+1)\kappa$. Thus $d_{\Omega}(0, z) < 18R_{m,\theta}(K+1)\kappa$. Therefore

$$\begin{aligned} d(X(1, e, z), e) &\leq d(X(1, e, z), X(1, e, w(N))) + d(X(1, e, w(N)), e) \\ &< 12R_{m,\theta}(K+1)\kappa F(K+18R_{m,\theta}(K+1)\kappa) + \varepsilon \left(1 - \frac{1}{n}(1 - \frac{1}{p})\right). \end{aligned} \quad (5.23)$$

Consequently if

$$\kappa < \min \left(\frac{1}{2}, \kappa(n, p, K, \varepsilon), \frac{\varepsilon}{12nqF(K+18R_{m,\theta}(K+1))R_{m,\theta}(K+1)} \right),$$

$d(X(1, e, z), e) < \varepsilon \left(1 - \frac{1}{n}(1 - \frac{1}{p} - \frac{1}{q})\right)$ holds. Now we set $p = q = 4$ and κ to be a positive number such that

$$\kappa < \min \left(\frac{\varepsilon}{48nF(K+18R_{m,\theta}(K+1))R_{m,\theta}(K+1)}, \frac{1}{2} \right). \quad (5.24)$$

For such a κ , it holds that if $w \in \mathcal{D}_{\varepsilon_n, K, N, \kappa}$ then $z \in \mathcal{D}_{\varepsilon_{2n}}$ for any $z \in U_{2\kappa}(w(N))$. That is, $w \in U_{\kappa}(w(N)) \subset U_{2\kappa}(w(N)) \subset \mathcal{D}_{\varepsilon_{2n}}$. Applying Lemma 5.2 (1) to the case where $\varphi_1 = w(N)$, $\varphi_2 = \varphi$, $r = \kappa$, $\delta = \sqrt{2} - 1, 1/3$, we have if

$$\|\varphi - w(N)\|_H < \frac{\kappa}{3(6\kappa + 1 + 2(K+2))}$$

then

$$w \in U_{\kappa}(w(N)) \subset U_{4\kappa/3}(\varphi) \subset U_{\sqrt{2}\kappa}(\varphi) \subset U_{2\kappa}(w(N)) \subset \mathcal{D}_{\varepsilon_{2n}}.$$

This completes the proof of (1) from which follow (2) and (3). \square

6 H -simply connected set in a Wiener space

We introduce the following notions.

Definition 6.1. Let D be an H -open and measurable subset of Ω with $\mu(D) > 0$. Here D is said to be H -open if for any $w \in D$, there exists $\varepsilon > 0$ such that $w + \{h \in H \mid \|h\|_H < \varepsilon\} \subset D$.

(1) D is called an H -connected set if, whenever $w, w+h \in D$, there exists a C^∞ curve $h : [0, 1] \rightarrow H$ such that $h(0) = 0$ and $h(1) = h$ and $w+h(\tau) \in D$ for all $0 \leq \tau \leq 1$.

(2) D is called an H -simply connected set if the following holds: Let us fix any point w of D . Let $\{h(0, \tau) \mid 0 \leq \tau \leq 1\}$ and $\{h(1, \tau) \mid 0 \leq \tau \leq 1\}$ be C^∞ curves on H such that $h(0, 0) = h(1, 0) = 0$, $h(0, 1) = h(1, 1)$ and $\{w+h(i, \tau) \mid 0 \leq \tau \leq 1\} \subset D$ for $i = 0, 1$. Then there exists a C^∞ map $\mathcal{H} : [0, 1]^2 \rightarrow H$ which may depend on w such that

- (i) $\mathcal{H}(0, \tau) = h(0, \tau)$, $\mathcal{H}(1, \tau) = h(1, \tau)$ for all $0 \leq \tau \leq 1$,
- (ii) $\mathcal{H}(\sigma, 0) = 0$ and $\mathcal{H}(\sigma, 1) = h(0, 1) = h(1, 1)$ for all σ ,
- (iii) $w + \mathcal{H}(\sigma, \tau) \in D$ holds for any $(\sigma, \tau) \in [0, 1]^2$.

The ball like set $U_r(\varphi)$ is H -connected. We need the following lemma to prove this statement. Also this lemma will be used in the proof of Proposition 6.5 (2).

Lemma 6.2. *Let $\varphi_i \in H$ and $r_i > 0$ ($i = 1, 2$). The following three conditions (i), (ii), (iii) are equivalent.*

- (i) $\mu(U_{r_1}(\varphi_1) \cap U_{r_2}(\varphi_2)) > 0$.
- (ii) $U_{r_1}(\varphi_1) \cap U_{r_2}(\varphi_2) \neq \emptyset$.
- (iii) $U_{r_1}(\varphi_1) \cap U_{r_2}(\varphi_2) \cap H \neq \emptyset$.

Proof. It is trivial that (i) implies (ii). The implication (ii) \implies (iii) follows from that $\lim_{N \rightarrow \infty} d_\Omega(w(N), w) = 0$ for any $w \in \Omega$. We prove (iii) implies (i). By the assumption, there exists $h \in U_{r_1}(\varphi_1) \cap U_{r_2}(\varphi_2) \cap H$. Let ε be a sufficiently small positive number. Let $w \in U_\varepsilon(0)$. Then $w + h \in U_{r_1}(\varphi_1) \cap U_{r_2}(\varphi_2)$ and $\mu(U_\varepsilon(0) + h) > 0$. This proves (i). \square

Lemma 6.3. *Let $D_i = U_{r_i}(\varphi_i)$ ($1 \leq i \leq n$). Assume that $(\cup_{i=1}^k D_i) \cap D_{k+1} \neq \emptyset$. Then $D = \cup_{i=1}^n D_i$ is an H -connected set.*

Proof. Clearly, D_i, D are H -open sets. Let $w, w + h \in D$. Without loss of generality, we may assume that $w \in D_1$, $w + h \in D_i$ and $D_k \cap D_{k+1} \neq \emptyset$ for all $1 \leq k \leq i-1$. Let $\psi_k \in D_k \cap D_{k+1} \cap H$. Let $\varphi_{k,w(N)^\perp} = \varphi_k + w(N)^\perp$ and $\psi_{k,w(N)^\perp} = \psi_k + w(N)^\perp$. Then for sufficiently large N , it holds that

$$\{(1-\tau)\varphi_{k,w(N)^\perp} + \tau\psi_{k,w(N)^\perp} \mid 0 \leq \tau \leq 1\} \subset D_k \quad (k = 1, \dots, i-1), \quad (6.1)$$

$$\{(1-\tau)\psi_{k-1,w(N)^\perp} + \tau\varphi_{k,w(N)^\perp} \mid 0 \leq \tau \leq 1\} \subset D_k \quad (k = 2, \dots, i) \quad (6.2)$$

$$\{(1-\tau)w + \tau\varphi_{1,w(N)^\perp} \mid 0 \leq \tau \leq 1\} \subset D_1, \quad (6.3)$$

$$\{(1-\tau)(w+h) + \tau\varphi_{i,w(N)^\perp} \mid 0 \leq \tau \leq 1\} \subset D_i. \quad (6.4)$$

This follows from Theorem 3.1. Hence, we have proved the existence of a piecewise linear path $h = h(\tau)$ ($0 \leq \tau \leq 1$) such that $h(0) = 0$, $h(1) = h$ and $w + h(\tau) \subset D$ for all $0 \leq \tau \leq 1$. Note that if $\sup_\tau \|\tilde{h}(\tau) - h(\tau)\|_H$ is sufficiently small, then $\{w + \tilde{h}(\tau) \mid 0 \leq \tau \leq i+1\} \subset D$. Thus we see the existence of a smooth path connecting w and $w + h$. \square

The space of mapping, $H^1([0, 1] \rightarrow G)$, is a C^∞ -Hilbert manifold naturally. In the lemma below, we use this differentiable structure.

Lemma 6.4. *Assume that G is a simply connected compact Lie group. Let V be an open set of G which is diffeomorphic to a ball in a Euclidean space. Let*

$$H_V^1 = \{\gamma \in H^1([0, 1] \rightarrow G) \mid \gamma_0 = e, \gamma_1 \in V\}.$$

Let $\{\gamma(i, \tau) \mid 0 \leq \tau \leq 1\} \subset H_V^1$ ($i = 0, 1$) be two C^∞ -curves with the same starting point and end point in H_V^1 , that is, we assume

$$\gamma(0, 0) = \gamma(1, 0) \in H_V^1, \gamma(0, 1) = \gamma(1, 1) \in H_V^1.$$

Then there exists a C^∞ -homotopy map $\mathcal{M} : (\sigma, \tau) \in [0, 1]^2 \mapsto \mathcal{M}(\sigma, \tau) \in H_V^1$ such that

- (i) $\mathcal{M}(0, \tau) = \gamma(0, \tau)$ and $\mathcal{M}(1, \tau) = \gamma(1, \tau)$ for all τ ,
- (ii) $\mathcal{M}(\sigma, 0) = \gamma(0, 0) = \gamma(1, 0)$ and $\mathcal{M}(\sigma, 1) = \gamma(0, 1) = \gamma(1, 1)$ for all σ .

Proof. This follows from that $\pi_2(G) = 0$ and so $\pi_1(L_e(G)) = 0$. See [8] and [32]. This is the result in continuous category. In the case of H^1 -paths, it suffices to approximate the continuous homotopy by a smooth homotopy. \square

Proposition 6.5. *Assume that G is a simply connected compact Lie group.*

- (1) *The subset \mathcal{D}_ε is an H -connected and H -simply connected set for sufficiently small ε .*
- (2) *Let $\{U_{4\kappa_i/3}(\varphi_i), i = 1, 2, \dots\}$ be the sets which are defined in Lemma 5.1 (3). Then if necessary, by changing the order of the sets, we have*

$$\mu\left(\left(\bigcup_{i=1}^n U_{4\kappa_i/3}(\varphi_i)\right) \cap U_{4\kappa_{n+1}/3}(\varphi_{n+1})\right) > 0 \quad \text{for all } n \geq 1.$$

Proof. (1) First we prove that \mathcal{D}_ε is an H -connected set. Assume that $w, w+h \in \mathcal{D}_\varepsilon$. Then $X(1, e, w+h), X(1, e, w) \in B_\varepsilon(e)$. Let $Z(t, h, w)$ be the H^1 -path in Proposition 3.7. Since $X(1, e, w+h) = X(1, Z(1, h, w), w)$, $t \mapsto Z(t, h, w)$ is a H^1 -curve on G starting at e and $Z(1, h, w) \in X^{-1}(1, \cdot, w)(B_\varepsilon(e))$. Also $e \in X^{-1}(1, \cdot, w)(B_\varepsilon(e))$ holds. Since G is simply connected and $X^{-1}(1, \cdot, w)(B_\varepsilon(e))$ is a contractive set, there exists a map $(\tau, t) \in [0, 1]^2 \mapsto \gamma^{h,w}(\tau)_t \in G$ such that

- (i) $\gamma^{h,w}(0)_t = e$ and $\gamma^{h,w}(1)_t = Z(t, h, w)$ for all $0 \leq t \leq 1$,
- (ii) $\tau \in [0, 1] \mapsto \gamma^{h,w}(\tau)$ is a C^∞ -map with values in $H_{X^{-1}(1, \cdot, w)(B_\varepsilon(e))}^1$.

Now we define $h(\tau) = \zeta(\gamma^{h,w}(\tau), w)$. See Proposition 3.7 for the definition of ζ . The mapping $\tau \in [0, 1] \mapsto h(\tau)$ is a C^∞ -curve on H . Also $X(t, \gamma^{h,w}(\tau)_t, w) = X(t, e, w+h(\tau))$ ($(\tau, t) \in [0, 1]^2$) holds by the definition. Therefore $h(0) = 0$, $h(1) = h$ and $X(1, e, w+h(\tau)) \in B_\varepsilon(e)$ for all $0 \leq \tau \leq 1$. This proves that \mathcal{D}_ε is an H -connected set. Next we prove the H -simply connectedness of \mathcal{D}_ε . Let $\tau \in [0, 1] \mapsto h(i, \tau) \in H$ ($i = 0, 1$) be C^∞ -curves on H such that

- (i) $w + h(i, \tau) \in \mathcal{D}_\varepsilon$ for all $0 \leq \tau \leq 1$ and $i = 0, 1$.
- (ii) $h(0, 0) = h(1, 0) = 0$, $h(0, 1) = h(1, 1)$.

Then $Z(t, h(0, 0), w) = Z(t, h(1, 0), w) = e$ and $Z(t, h(0, 1), w) = Z(t, h(1, 1), w)$ hold for all $0 \leq t \leq 1$. Also $t \mapsto Z(t, h(i, \tau), w)$ is a H^1 -curve on G starting at e and the end point $Z(1, h(i, \tau), w) \in X^{-1}(1, \cdot, w)(B_\varepsilon(e))$ for all $0 \leq \tau \leq 1$ and $i = 0, 1$. Therefore $\tau \mapsto Z(\cdot, h(i, \tau), w)$ is a C^1 -map from $[0, 1]$ to $H_{X^{-1}(1, \cdot, w)(B_\varepsilon(e))}^1$. Since $B_\varepsilon(e)$ is a contractive set, $H_{X^{-1}(1, \cdot, w)(B_\varepsilon(e))}^1$ is also a simply connected set by Lemma 6.4. Therefore there exists a C^∞ homotopy map

$$(\sigma, \tau) \in [0, 1]^2 \mapsto \mathcal{M}^{h,w}(\sigma, \tau) \in H_{X^{-1}(1, \cdot, w)(B_\varepsilon(e))}^1 \quad (6.5)$$

such that

- (i) $\mathcal{M}^{h,w}(i, \tau)_t = Z(t, h(i, \tau), w)$ for all $0 \leq \tau, t \leq 1$ and $i = 0, 1$,
- (ii) $\mathcal{M}^{h,w}(\sigma, 0)_t = Z(t, h(0, 0), w) = Z(t, h(1, 0), w) = e$ and $\mathcal{M}^{h,w}(\sigma, 1)_t = Z(t, h(0, 1), w) = Z(t, h(1, 1), w)$ for all $0 \leq \sigma \leq 1$.

Let

$$\mathcal{H}(\sigma, \tau) = \zeta \left(\mathcal{M}^{h,w}(\sigma, \tau), w \right). \quad (6.6)$$

Then

- (i) $\mathcal{H}(i, \tau) = h(i, \tau)$ for all $0 \leq \tau \leq 1$ and $i = 0, 1$,
- (ii) For all σ , $\mathcal{H}(\sigma, 0) = 0$ and $\mathcal{H}(\sigma, 1) = h(0, 1) = h(1, 1)$,
- (iii) The mapping $(\sigma, \tau) \in [0, 1]^2 \mapsto \mathcal{H}(\sigma, \tau) \in H$ is C^∞ ,
- (iv) $w + \mathcal{H}(\sigma, \tau) \in \mathcal{D}_\varepsilon$ for all (σ, τ) .

These complete the proof.

(2) Since the map $h(\in H) \mapsto X(\cdot, e, h)(\in H^1([0, 1] \rightarrow G \mid \gamma(0) = e))$ is a diffeomorphism, $\mathcal{D}_\varepsilon \cap H$ is diffeomorphic to $H_{B_\varepsilon(e)}^1$. Hence, $\mathcal{D}_\varepsilon \cap H$ is an open connected subset of H . Since $U_{4\kappa_i/3}(\varphi_i) \cap H$ is an open subset of H and $\mathcal{D}_\varepsilon \cap H = \cup_{i=1}^\infty (U_{4\kappa_i/3}(\varphi_i) \cap H)$, it is an easy exercise to show that if necessary, by changing the order of the sets, we have

$$\cup_{i=1}^n (U_{4\kappa_i/3}(\varphi_i) \cap H) \cap U_{4\kappa_{n+1}/3}(\varphi_{n+1}) \neq \emptyset \quad \text{for all } n = 1, 2, \dots$$

Thus, by Lemma 6.2, we complete the proof. \square

Lemma 6.6 (Stokes theorem in H -direction). (1) Let $f \in \mathbb{D}^{1,q}(W^d)$, where $q > 1$. Then for any C^1 -curve $h = h(\tau)$ ($0 \leq \tau \leq 1$) on H , we have

$$f(w + h(1)) = f(w + h(0)) + \int_0^1 \left((Df)(w + h(t)), \dot{h}(t) \right)_H dt \quad \mu\text{-almost all } w. \quad (6.7)$$

(2) Let $\beta \in \mathbb{D}^{1,q}(W^d, H^*)$, where $q > 1$. Let $\mathcal{H} = \mathcal{H}(\sigma, \tau)$ ($(\sigma, \tau) \in [0, 1]^2$) be a C^2 -map with values in H . We assume that $\mathcal{H}(\sigma, 0) = \mathcal{H}(0, 0)$ and $\mathcal{H}(\sigma, 1) = \mathcal{H}(0, 1)$ for all $0 \leq \sigma \leq 1$. Then it holds that

$$\begin{aligned} & \int_0^1 (\beta(w + \mathcal{H}(1, \tau)), \partial_\tau \mathcal{H}(1, \tau)) d\tau - \int_0^1 (\beta(w + \mathcal{H}(0, \tau)), \partial_\tau \mathcal{H}(0, \tau)) d\tau \\ &= \iint_{(\sigma, \tau) \in [0, 1]^2} (d\beta)(w + \mathcal{H}(\sigma, \tau)) (\partial_\sigma \mathcal{H}(\sigma, \tau), \partial_\tau \mathcal{H}(\sigma, \tau)) d\sigma d\tau \quad \mu\text{-almost all } w. \end{aligned} \quad (6.8)$$

Proof. (1) This is trivial for $f \in \mathfrak{F}C_b^\infty(W^d)$. General cases follow from a limiting argument.

(2) First we assume that $\beta \in \mathfrak{F}C_b^\infty(W^d, H^*)$. By the definition of the exterior differential, we have

$$d\beta(w)(X, Y) = ((D\beta)(w)[X], Y) - ((D\beta)(w)[Y], X),$$

where $X, Y \in H$. Here $(D\beta)(w)[X]$ denotes the derivative in the direction to X . Let $\phi(\sigma) = \int_0^1 (\beta(w + \mathcal{H}(\sigma, \tau)), \partial_\tau \mathcal{H}(\sigma, \tau)) d\tau$. We have

$$\begin{aligned} \dot{\phi}(\sigma) &= \int_0^1 ((D\beta)(w + \mathcal{H}(\sigma, \tau))[\partial_\sigma \mathcal{H}(\sigma, \tau)], \partial_\tau \mathcal{H}(\sigma, \tau)) d\tau + \int_0^1 (\beta(w + \mathcal{H}(\sigma, \tau)), \partial_\sigma \partial_\tau \mathcal{H}(\sigma, \tau)) d\tau \\ &= \int_0^1 (d\beta)(w + \mathcal{H}(\sigma, \tau)) (\partial_\sigma \mathcal{H}(\sigma, \tau), \partial_\tau \mathcal{H}(\sigma, \tau)) d\tau \\ &\quad + \int_0^1 ((D\beta)(w + \mathcal{H}(\sigma, \tau))[\partial_\tau \mathcal{H}(\sigma, \tau)], \partial_\sigma \mathcal{H}(\sigma, \tau)) d\tau + \int_0^1 (\beta(w + \mathcal{H}(\sigma, \tau)), \partial_\sigma \partial_\tau \mathcal{H}(\sigma, \tau)) d\tau \end{aligned}$$

and

$$\begin{aligned} \int_0^1 ((D\beta)(w + \mathcal{H}(\sigma, \tau))[\partial_\tau \mathcal{H}(\sigma, \tau)], \partial_\sigma \mathcal{H}(\sigma, \tau)) d\tau + \int_0^1 (\beta(w + \mathcal{H}(\sigma, \tau)), \partial_\sigma \partial_\tau \mathcal{H}(\sigma, \tau)) d\tau \\ = (\beta(w + \mathcal{H}(\sigma, 1)), \partial_\sigma \mathcal{H}(\sigma, 1)) - (\beta(w + \mathcal{H}(\sigma, 0)), \partial_\sigma \mathcal{H}(\sigma, 0)) = 0. \end{aligned}$$

Therefore we get

$$\phi(1) - \phi(0) = \iint_{(\sigma, \tau) \in [0, 1]^2} (d\beta)(w + \mathcal{H}(\sigma, \tau)) (\partial_\sigma \mathcal{H}(\sigma, \tau), \partial_\tau \mathcal{H}(\sigma, \tau)) d\sigma d\tau. \quad (6.9)$$

By the limiting argument, we complete the proof. \square

7 A retraction map in a Wiener space

Let $X(t, a, w)$ be the solution of the SDE which is defined in Proposition 3.7. In this section, we construct a retraction map from a tubular neighborhood of the submanifold S to S . Recall that S is defined by

$$S = \{w \in \Omega \mid X(1, e, w) = e\}.$$

By Proposition 3.7, it is easy to see that $w \mapsto X(t, e, w)$ is H -differentiable map and

$$(R_{X(t, e, w)})_*^{-1} DX(t, e, w)[h] = \int_0^t Ad(X(s, e, w)) \dot{h}(s) ds.$$

Note that the differential form $\alpha \in \mathbb{D}^{k, q}(\wedge^p T^* L_e(G))$ is a measurable map from $L_e(G)$ to $\wedge^p H_0^*$. For $\alpha \in \mathbb{D}^{k, q}(\wedge^p T^* L_e(G))$, define the pull-back of α by X as follows:

$$(X^* \alpha)(w) = \alpha(X(w))(U(w), \dots, U(w)),$$

where $U(w)h = \int_0^t Ad(X(s, e, w)) \dot{h}(s) ds$. Since $X_* \mu_e = \nu_e$, $X^* \alpha \in L^p(\wedge^p T^* S)$. In fact, the map X^* gives isomorphisms between Sobolev spaces as follows.

Proposition 7.1. (1) *Let k be a non-negative integer and $q > 1$. The mapping X^* is a bijective linear isometry from $\mathbb{D}^{k, q}(\wedge^p T^* L_e(G))$ to $\mathbb{D}^{k, q}(\wedge^p T^* S)$.*
(2) *For any $\alpha \in \mathbb{D}^{k, q}(\wedge^p T^* L_e(G))$, we have $d_S X^* \alpha = X^* d\alpha$.*

Proof. (1) The surjectivity follows from the denseness of $X^* \mathfrak{F}C_b^\infty(L_e(G))$ in $\mathbb{D}^\infty(S)$. See Lemma 3.3 in [2]. In the case of tensors, the proof of the bijectivity can be found in Proposition 3.6 in [2]. The same proof works in the case of differential forms.

(2) This follows from a direct calculation. \square

Let ε be a sufficiently small positive number. For $a \in B_\varepsilon(e)$, let

$$\psi_\varepsilon(a, w) = - \int_0^1 Ad(X(s, e, w)^{-1}) (\log a) ds \in H.$$

Here \log is the inverse mapping of $\exp : \mathfrak{g} \rightarrow G$. Using this, we define

$$\Psi_\varepsilon(w) = w + \psi_\varepsilon(X(1, e, w), w) \quad w \in \mathcal{D}_\varepsilon. \quad (7.1)$$

By Proposition 3.7, $\Psi_\varepsilon(w) \in S$ for all $w \in \mathcal{D}_\varepsilon$. Note that $\sup_{w \in \Omega} \|D\Psi_\varepsilon(w)\|_{L(H,H)} < \infty$. We define the pull-back of $\theta \in \mathfrak{F}C_b^\infty(\wedge^p T^*S)$ by Ψ_ε as follows:

$$(\Psi_\varepsilon^* \theta)(w) = \theta(\Psi_\varepsilon(w))(D\Psi_\varepsilon(w), \dots, D\Psi_\varepsilon(w)).$$

The statement (5) in the following proposition which follows from the result in rough path analysis is important in the proof of our main results.

Proposition 7.2. (1) *Let $q > 1$. For any $\eta \in \mathbb{D}^\infty(W^d)$, it holds that*

$$\begin{aligned} & \int_{\mathcal{D}_\varepsilon} |\Psi_\varepsilon^* \theta(w)|^q \eta(w) d\mu(w) \\ &= \int_{B_\varepsilon(e)} da \int_S d\mu_e(w) |\theta(w) ((D\Psi_\varepsilon)(w + \psi_\varepsilon(a^{-1}, w)), \dots, (D\Psi_\varepsilon)(w + \psi_\varepsilon(a^{-1}, w)))|^q \\ & \quad \times \eta(w + \psi_\varepsilon(a^{-1}, w)) \exp\left(-(\log a, b(1, w)) - \frac{1}{2} |\log a|^2\right), \end{aligned} \quad (7.2)$$

where $b(1, w) = \int_0^1 \text{Ad}(X(t, e, w)) \circ dw(t)$. In particular $\|\Psi_\varepsilon^* \theta\|_{L^q(\mathcal{D}_\varepsilon, \mu)} \leq C_{q,r} \|\theta\|_{L^r(S, \mu_e)}$ for any $1 < q < r$.

(2) *Let χ be a smooth function on \mathbb{R} such that $\chi = 1$ in a neighborhood of 0 and $\text{supp } \chi \subset (-\infty, \varepsilon^2)$. Set $\hat{\chi}(w) = \chi(d(X(1, e, w), e)^2)$. Define $T_{\chi, \varepsilon} \theta = \hat{\chi} \Psi_\varepsilon^* \theta$ for $\theta \in \mathfrak{F}C_b^\infty(\wedge^p T^*S)$. Then $T_{\chi, \varepsilon}$ can be extended uniquely to a bounded linear operator from $\mathbb{D}^{k,r}(\wedge^p T^*S)$ to $\mathbb{D}^{k,q}(\wedge^p H^*)$ for any $1 < q < r$ and $k \in \mathbb{N} \cup \{0\}$. Moreover it holds that*

$$dT_{\chi, \varepsilon} \theta = d\hat{\chi} \wedge \Psi_\varepsilon^* \theta + \hat{\chi} \Psi_\varepsilon^* d_S \theta. \quad (7.3)$$

(3) *The pull-back $\iota^* \beta \in \mathbb{D}^{k,q}(\wedge^p T^*S)$ is well-defined for p -form β on W^d with $\|\beta\|_{k,r} < \infty$ for sufficiently large k and any $1 < q < r$. Moreover it holds that*

$$d_S \iota^* \beta = \iota^* d\beta. \quad (7.4)$$

(4) *For sufficiently large k and $q > 1$, it holds that for any $\theta \in \mathbb{D}^{k,q}(\wedge^p T^*S)$*

$$\iota^* T_{\chi, \varepsilon} \theta = \theta. \quad (7.5)$$

(5) *Let $\varphi \in H$ and $U_r(\varphi) \subset \mathcal{D}_\varepsilon$. Then there exists a constant C which depends only on r, φ, ε such that*

$$\|\Psi_\varepsilon^* \theta\|_{L^2(U_r(\varphi))} \leq C \|\theta\|_{L^2(\mu_e)}. \quad (7.6)$$

Proof. Noting that $X(t, e, w + \psi_\varepsilon(a, w)) = e^{-t \log a} X(t, e, w)$, (1) follows from the quasi-invariance of ν_e . See [18]. The extension property of (2) follows from (1). One can check the identity (7.3) by a direct calculation when θ is a smooth cylindrical form. General cases follow from an approximation argument. Part (3) is easy to check when β is a smooth cylindrical form. General cases follows from a limiting argument. Part (4) follows from $D\Psi_\varepsilon(w) = P(w)$ on S , where $P(w)$ is a projection operator from H onto the tangent space of S at w . Part (5) follows from (1) and Proposition 3.7 (2). \square

8 Proof of the main theorem

The following immediate consequence of the ergodicity of the Wiener measure under translations by H is used to construct f in Theorem 2.1 by the local data on $U_r(\varphi)$.

Lemma 8.1. *Let A, B be measurable subsets of W^d with $\mu(A) > 0$ and $\mu(B) > 0$. Then there exists $h \in H$ and a measurable subset $A_0 \subset A$ such that $\mu(A_0) > 0$ and $A_0 + h \subset B$.*

Let χ be a smooth nonnegative function such that $\chi(u) = 1$ for $u \leq 4\varepsilon^2/9$ and $\chi(u) = 0$ for $u \geq 9\varepsilon^2/16$. Let $\hat{\chi}(w) = \chi(d(X(1, e, w), e)^2)$.

Lemma 8.2. *Let θ be the 1-form on S in Theorem 2.1. Let ε be a sufficiently small positive number. Let $\beta = \Psi_\varepsilon^* \theta$. Let $1 < q < p$. Then there exists a measurable function F on \mathcal{D}_ε and ρ_n ($n \in \mathbb{N}$) on Ω such that the following hold.*

- (1) *The function ρ_n is a bounded non-negative ∞ -quasi-continuous function and $\rho_n \in \mathbb{D}^\infty(W^d)$ holds.*
- (2) *For any $r > 1$ and $k \in \mathbb{N}$, $\lim_{n \rightarrow \infty} C_r^k(\{w \in \Omega \mid \rho_n(w) = 1\}^c) = 0$ and $\lim_{n \rightarrow \infty} \|\rho_n - 1\|_{r,k} = 0$.*
- (3) *There exists $F_n \in \mathbb{D}^{1,2}(W^d) \cap \mathbb{D}^{\infty,q}(W^d)$ such that $F(w) = F_n(w)$ and $dF_n(w) = \beta(w)$ for μ -almost all w of $\{w \in \Omega \mid \rho_n(w) \neq 0\} \cap \mathcal{D}_{\varepsilon/2}$.*
- (4) *Let $\hat{F}_n = \tilde{F}_n \rho_n \hat{\chi}$, where \tilde{F}_n is a (q, ∞) -quasi-continuous version of F_n . It holds that $\hat{F}_n \in \mathbb{D}^{1,2}(W^d) \cap \mathbb{D}^{\infty,q}(W^d)$ and*

$$d\hat{F}_n = \beta \rho_n \hat{\chi} + \tilde{F}_n d\rho_n \hat{\chi} + \tilde{F}_n \rho_n d\hat{\chi}. \quad (8.1)$$

Proof. Let χ_0 be a smooth decreasing function on \mathbb{R} such that $\chi_0(u) = 1$ for $u \leq 9\varepsilon^2/4$ and $\text{supp } \chi_0 \subset (-\infty, 4\varepsilon^2)$. Let $\gamma = T_{\chi_0, 2\varepsilon} \theta$. Then $\gamma \in \mathbb{D}^{\infty,q}(W^d, H^*)$. Also note that $\gamma = \beta$ and $d\gamma = 0$ on \mathcal{D}_ε . The latter result follows from Proposition 7.2 (2). Let $U_{\sqrt{2}\kappa_i}(\varphi_i)$ ($i = 1, 2, \dots$) be the covering of \mathcal{D}_ε in Lemma 5.1 (3) and Proposition 6.5 (2). Let us choose r_i such that $4\kappa_i/3 < r_i < \sqrt{2}\kappa_i$. Since $d\gamma = 0$ on $U_{\sqrt{2}\kappa_i}(\varphi_i)$ and $\gamma \in L^2(U_{\sqrt{2}\kappa_i}(\varphi_i))$, by Theorem 4.7, we see that there exist $g_i \in \mathbb{D}^{\infty,q}(W^d) \cap \mathbb{D}^{1,2}(W^d)$ such that $dg_i = \gamma$ on $U_{r_i}(\varphi_i)$. However g_i on $U_{r_i}(\varphi_i)$ is not determined uniquely, in fact, there is an ambiguity of additive constant. Actually we prove that there are constants c_i and a measurable function F on \mathcal{D}_ε such that $F(w) = g_i(w) + c_i$ almost all $w \in U_{r_i}(\varphi_i)$ for any i and r_i . First set $c_1 = 0$. We define c_i ($i \geq 2$) inductively in the following way. Suppose that there exist c_1, \dots, c_i and a measurable function G_i on $\cup_{j=1}^i U_{r_j}(\varphi_j)$ such that $G_i(w) = g_j(w) + c_j$ almost all $w \in U_{r_j}(\varphi_j)$ for all $1 \leq j \leq i$. By Theorem 4.7, there exist $G_{i,j} \in \mathbb{D}^{1,2}(W^d) \cap \mathbb{D}^{\infty,q}(W^d)$ such that $G_{i,j}(w) = G_i(w)$ on $U_{r_j}(\varphi_j)$. We prove that for any $\{r'_j\}$ with $4\kappa_j/3 < r'_j < r_j$ ($1 \leq j \leq i$) there exists $H_i \in \mathbb{D}^{1,2}(W^d) \cap \mathbb{D}^{\infty,q}(W^d)$ such that $H_i = G_i$ and $dH_i = \beta$ on $\cup_{j=1}^i U_{r'_j}(\varphi_j)$.

Note that there exist $\phi_j \in \mathbb{D}^\infty(W^d)$ ($1 \leq j \leq i+2$) such that the following identity holds. For $1 \leq j \leq i$

$$\phi_j(w) = \begin{cases} 1 & w \in U_{r'_j + \varepsilon_j}(\varphi_j), \\ 0 & w \in U_{r'_j + \varepsilon'_j}(\varphi_j)^c \end{cases}$$

and

$$\phi_{i+1}(w) = \begin{cases} 0 & w \in \cup_{j=1}^i U_{r'_j + \varepsilon_j - \delta'_j}(\varphi_j), \\ 1 & w \in \left(\cup_{j=1}^i U_{r'_j + \varepsilon_j - \delta_j}(\varphi_j) \right)^c, \end{cases}$$

$$\phi_{i+2}(w) = \begin{cases} 1 & w \in \cup_{j=1}^i U_{r'_j+\varepsilon_j-\delta'_j-\tau'_j}(\varphi_j), \\ 0 & w \in \left(\cup_{j=1}^i U_{r'_j+\varepsilon_j-\delta'_j-\tau'_j}(\varphi_j) \right)^c. \end{cases}$$

Here we choose positive numbers such that $0 < \delta_j < \delta'_j < \varepsilon_j < \varepsilon'_j$, $\varepsilon_j - \delta'_j - \tau'_j > 0$, $0 < \tau_j < \tau'_j$ and $r'_j + \varepsilon'_j < r_j$. These functions can be constructed explicitly in a similar way to $\tilde{\rho}(w)$ in the proof of Theorem 4.7 using mollifiers. Since $\sum_{j=1}^{i+1} \phi_j(w) \geq 1$ for any $w \in \Omega$,

$$\tilde{\phi}_j(w) = \frac{\phi_j(w)}{\sum_{j=1}^{i+1} \phi_j(w)}$$

belongs to $\mathbb{D}^{1,2}(W^d) \cap \mathbb{D}^{\infty,q}(W^d)$ and $\sum_{j=1}^{i+1} \tilde{\phi}_j(w) = 1$ for all $w \in \Omega$. This is a partition of unity associated with the covering of Ω :

$$U_{r'_j+\varepsilon'_j}(\varphi_j) \ (1 \leq j \leq i), \quad \left(\cup_{j=1}^i U_{r'_j+\varepsilon_j-\delta'_j}(\varphi_j) \right)^c$$

Since $\phi_{i+2}(w)\phi_{i+1}(w) = 0$ for all $w \in \Omega$, we have

$$\begin{aligned} G_i(w)\phi_{i+2}(w) &= \sum_{j=1}^{i+1} G_i(w)\phi_{i+2}(w)\tilde{\phi}_j(w) \\ &= \sum_{j=1}^i G_{i,j}(w)\phi_{i+2}(w)\tilde{\phi}_j(w). \end{aligned} \tag{8.2}$$

Therefore $H_i = G_i\phi_{i+2}$ is the desired function.

By using the existence of H_i and the H -simply connectedness of \mathcal{D}_ε , we next prove the existence of a measurable function G_{i+1} on $\cup_{j=1}^{i+1} U_{r'_j}(\varphi_j)$ and a constant c_{i+1} such that $G_{i+1}(w) = G_i(w)$ for almost all $w \in \cup_{j=1}^i U_{r'_j}(\varphi_j)$ and $G_{i+1}(w) = g_{i+1}(w) + c_{i+1}$ for almost all $w \in U_{r'_{i+1}}(\varphi_{i+1})$. Since $\mu\left(\left(\cup_{j=1}^i U_{r'_j}(\varphi_j)\right) \cap U_{r'_{i+1}}(\varphi_{i+1})\right) > 0$, there exists a piecewise linear path $\varphi \in H$, $\delta > 0$ and $1 \leq i_0 \leq i$ such that $U_\delta(\varphi) \subset U_{r'_{i+1}}(\varphi_{i+1}) \cap U_{r'_{i_0}}(\varphi_{i_0})$. Because $d(g_{i+1} - g_{i_0}) = 0$ on $U_\delta(\varphi)$, $g_{i+1}(w) - g_{i_0}(w)$ is equal to a constant almost all w on $U_\delta(\varphi)$. We choose c_{i+1} such that $g_{i+1}(w) + c_{i+1} = g_{i_0}(w) + c_{i_0} (= G_i(w))$ almost all $w \in U_\delta(\varphi)$. It suffices to prove that

$$g_{i+1}(w) + c_{i+1} = G_i(w) \quad \text{for almost all } w \in \left(\cup_{j=1}^i U_{r'_j}(\varphi_j) \right) \cap U_{r'_{i+1}}(\varphi_{i+1}). \tag{8.3}$$

Suppose that there exists a set $B \subset U_{r'_{i_1}}(\varphi_{i_1}) \cap U_{r'_{i+1}}(\varphi_{i+1})$ of positive measure for some $1 \leq i_1 \leq i$ and $c' > 0$ such that

$$|g_{i+1}(w) + c_{i+1} - G_i(w)| > c' \quad \text{for all } w \in B.$$

By the ergodicity of the Wiener measure, there exists a subset $A \subset U_\delta(\varphi)$ with positive measure and $h \in H$ such that $A + h \subset B$. Choose a point $\eta \in A$ such that $\mu(V_r(\eta) \cap A) > 0$ for all $r > 0$, where $V_r(\eta)$ is defined by (5.11). By the H -connectivity of $\cup_{j=1}^i U_{r'_j}(\varphi_j)$ and $U_{r_{i+1}}(\varphi_{i+1})$, there exists two C^∞ -curves $h(i, \tau)$ ($0 \leq \tau \leq 1$) on H such that $h(i, 0) = 0$, $h(i, 1) = h$ ($i = 0, 1$) and

$\eta + h(0, \tau) \subset \cup_{j=1}^i U_{r'_j}(\varphi_j)$ $\eta + h(1, \tau) \subset U_{r'_{i+1}}(\varphi_{i+1})$ for all $0 \leq \tau \leq 1$. By choosing δ to be a sufficiently small positive number, we have for all $0 \leq \tau \leq 1$,

$$V_\delta(\eta) + h(0, \tau) \subset \cup_{j=1}^i U_{r'_i}(\varphi_i) \quad (8.4)$$

$$V_\delta(\eta) + h(1, \tau) \subset U_{r'_{i+1}}(\varphi_{i+1}) \quad (8.5)$$

By the H -simply connectedness of \mathcal{D}_ε , there exists a C^∞ -map $\mathcal{H} = \mathcal{H}(\sigma, \tau)$ ($0 \leq \sigma, \tau \leq 1$) such that $\mathcal{H}(0, \tau) = h(0, \tau)$, $\mathcal{H}(1, \tau) = h(1, \tau)$ and $\eta + \mathcal{H}(\sigma, \tau) \subset \mathcal{D}_\varepsilon$ for all $(\sigma, \tau) \in [0, 1]^2$. Using the continuity of $X(1, e, \cdot)$ in the topology of d_Ω , we see that there exists $0 < \delta' < \delta$ such that for all $0 \leq \sigma, \tau \leq 1$ $V_{\delta'}(\eta) + \mathcal{H}(\sigma, \tau) \subset \mathcal{D}_\varepsilon$. Note that $dg_{i+1} = \beta$ on $U_{r'_{i+1}}(\varphi_{i+1})$ and $dH_i = \beta$ on $\cup_{j=1}^i U_{r'_j}(\varphi_j)$. By applying Lemma 6.6 and noting that $d\beta = 0$ on \mathcal{D}_ε , we obtain

$$(g_{i+1}(w + h) + c_{i+1}) - (g_{i+1}(w) + c_{i+1}) = G_i(w + h) - G_i(w) \quad \text{for almost all } w \in A \cap V_{\delta'}(\eta).$$

This is a contradiction. This implies (8.3). Inductively, we obtain a measurable function F on \mathcal{D}_ε such that for any i $F(w) = g_i(w) + c_i$ for some c_i and there exists $H_i \in \mathbb{D}^{1,2}(W^d) \cap \mathbb{D}^{\infty,q}(W^d)$ such that $F(w) = H_i(w)$ for almost all $w \in \cup_{j=1}^i U_{r'_j}(\varphi_j)$. Let χ_1 be a non-negative smooth non-increasing function such that $\chi_1(u) = 1$ for $u \leq (1/2)^m$ and $\chi_1(u) = 0$ for $u \geq (2/3)^m$. Let

$$\begin{aligned} \chi_{n,2}(w) &= \chi_1 \left(n^{-m} \left(\sum_{1 \leq i, j \leq d} \|C(w^i, w^j)\|_{m, \theta}^m + \sum_{1 \leq k \leq d} \|w^k\|_{m, \theta'/2}^m \right) \right), \\ \chi_{\kappa, N, 3}(w) &= \chi_1 \left(\kappa^{-m} \left(\sum_{k=1}^n \|w(N)^{\perp, k}\|_{m, \theta'/2}^m + \sum_{1 \leq i < j \leq d} \|C(w(N)^{\perp, i}, w(N)^{\perp, j})\|_{m, \theta}^m \right. \right. \\ &\quad \left. \left. + \sum_{1 \leq i \leq j \leq d} \|C(w(N)^i, w(N)^{\perp, j})\|_{m, \theta}^m + \sum_{1 \leq i \leq j \leq d} \|C(w(N)^{\perp, i}, w(N)^j)\|_{m, \theta}^m \right) \right), \end{aligned}$$

and set $\chi_{n, \kappa, N, 4}(w) = \chi_{n, 2}(w) \chi_{\kappa, N, 3}(w)$. Then we have $\{\chi_{n, \kappa, N, 4}(w) \neq 0\} \cap \mathcal{D}_{\varepsilon_2} \subset \mathcal{D}_{\varepsilon_2, n, N, \kappa}$. Now choosing $\kappa = \kappa(n)$ to be sufficiently small according to n as in Lemma 5.1, we have for sufficiently large $L_0 \in \mathbb{N}$,

$$\mathcal{D}_{\varepsilon_2, n, N, \kappa(n)} \subset \cup_{i=1}^{L_0} U_{4\kappa_i/3}(\varphi_i).$$

Therefore letting $N = a(\kappa(n))$ to be a sufficiently large natural number according to $\kappa = \kappa(n)$, we see that $\rho_n(w) = \chi_{n, \kappa(n), a(\kappa(n)), 4}(w)$ satisfies the properties (1), (2). As for (3), it suffices to set $F_n = H_i$ for sufficiently large i . Part (4) follows from (3). \square

We now can prove the main theorems.

Proof of Theorem 2.1. Let ρ_n be the function in Lemma 8.2. Then (1) holds. Let $f_n = \tilde{F}_n$. We construct f on S . Let $C_n = \{\rho_n \neq 0\} \cap \mathcal{D}_{\varepsilon/2}$. By Lemma 8.2 (2), $\lim_{n \rightarrow \infty} \mu_e(C_n^c) = 0$. For $n, n' \in \mathbb{N}$, we have

$$\tilde{F}_n(w) = \tilde{F}_{n'}(w) = F(w) \quad \text{for } \mu\text{-almost all } w \text{ of } C_n \cap C_{n'}. \quad (8.6)$$

Hence there exists a Borel measurable subset $B_{n, n'}$ such that $C_q^k(B_{n, n'}) = 0$ and

$$\tilde{F}_n(w) = \tilde{F}_{n'}(w) \quad \text{for all } w \in C_n \cap C_{n'} \cap B_{n, n'}^c. \quad (8.7)$$

This implies that $\tilde{F}_n(w) = \tilde{F}_{n'}(w)$ for μ_e -almost all $w \in C_n \cap C_{n'} \cap S$. Therefore there exists a measurable function f on S

$$f(w) = \tilde{F}_n(w) \quad \text{for } \mu_e\text{-almost all } w \in C_n \cap S. \quad (8.8)$$

For this f and f_n , (2) (i), (ii) holds. We prove (ii). Lemma 8.2 (3) shows that $dF_n = \beta = T_{\chi_0, 2\varepsilon}$ on C_n . Hence, using Proposition 7.2 (3) and (4), we can conclude that $d_S(\iota^* F_n) = \theta$ on $\{\rho_n \neq 0\} \cap S$ which implies $d_S f_n = \theta$ on $\{\rho_n \neq 0\} \cap S$. We prove (2) (iii). Note that $f\rho_n\eta = f_n\rho_n\eta \in \mathbb{D}^{\infty, q-}(W^d)$. Hence by Theorem 4.3 in [37], we have $f\rho_n\eta \in L^1(S, \mu_e)$. The equation in (2) (iv) is equivalent to

$$\int_S f_n \rho_n d_S^*(\rho_n \eta) d\mu_e = \int_S (d_S(f_n \rho_n), \rho_n \eta) d\mu_e$$

which follows from the integration by parts formula on S . We prove (2) (v). By the integration by parts formula on S , we have

$$\int_S \psi'_K(\hat{F}_n(w)) (d_S \hat{F}_n(w), \eta(w)) d\mu_e(w) = \int_S \psi_K(\hat{F}_n(w)) d_S^* \eta(w) d\mu_e(w). \quad (8.9)$$

By Lemma 8.2 (4), we get

$$d_S \hat{F}_n = \theta \rho_n + \tilde{F}_n d\rho_n. \quad (8.10)$$

Substituting (8.10) into (8.9) and replacing η by $\rho_n \eta$, we have

$$\begin{aligned} \int_S \psi'_K(f(w)\rho_n(w)) (\theta(w)\rho_n(w) + f(w)d\rho_n(w), \rho_n(w)\eta(w)) d\mu_e(w) \\ = \int_S \psi_K(f(w)\rho_n(w)) d_S^*(\rho_n \eta)(w) d\mu_e(w). \end{aligned} \quad (8.11)$$

Here we have used that $f(w) = \tilde{F}_n(w)$ μ_e -almost all w on $\{\rho_n \neq 0\}$. Letting $n \rightarrow \infty$, we obtain

$$\int_S \psi'_K(f(w)) (\theta(w), \eta(w)) d\mu_e(w) = \int_S \psi_K(f(w)) d_S^* \eta(w) d\mu_e(w). \quad (8.12)$$

This implies that the weak derivative of $\psi_K(f)$ is $\psi'_K(f)\theta$. Since $(d_S^* d_S, \mathfrak{F}C_b^\infty(W^d))$ is essentially self-adjoint (see [1], [2]), $\psi_K(f) \in \mathbb{D}^{1,2}(S)$ and $d_S \psi_K(f) = \psi'_K(f)\theta$. \square

We prove Theorem 2.2.

Proof of Theorem 2.2. Let $\bar{\alpha} = X^* \alpha$. Then $\bar{\alpha} \in L^2(\wedge^1 T^* S) \cap \mathbb{D}^{\infty, p}(\wedge^1 T^* S)$ and $d_S \bar{\alpha} = 0$ on S . By Theorem 2.1, there exists a measurable function g on S such that $d_S g = \bar{\alpha}$. By using Proposition 7.1 (1), we see that there exists a measurable function f on $L_e(G)$ such that $X^* f = g$ for μ_e -almost all w . Hence $X^* f^K = g^K$. By Proposition 7.1 and Theorem 2.1, we have $f^K \in \mathbb{D}^{1,2}(L_e(G))$ and $df^K = \psi'_K(f)\alpha$ which proves (1). Since $df^K = \psi'_K(f)\alpha$, using a similar argument to the proof of Lemma 14 in [3], we have

$$f^K(e^{\varepsilon h} \gamma) - f^K(\gamma) = \int_0^\varepsilon (\psi'_K(f(\gamma))\alpha(e^{sh} \gamma), h) ds.$$

Letting $K \rightarrow \infty$, we complete the proof of (2). Part (3) follows from (2). \square

We need the Weitzenböck formula for \square to prove Theorem 2.4. It will be proved below.

Lemma 8.3. *Let $C = \sum_{i=1}^d (\text{ad} \varepsilon_i)^2$, where $\{\varepsilon_i\}$ denotes an orthonormal system of \mathfrak{g} . Then*

$$\begin{aligned} (\square \alpha, h) &= (\nabla_{\nu_e}^* \nabla \alpha + \alpha + T_{b(1)} \alpha, h) \\ &\quad + \int_0^1 ((C\alpha)_t, h_t) dt - \int_0^1 \int_0^1 (C\alpha_t, h_s) dt ds, \end{aligned} \quad (8.13)$$

where $(T_v \alpha)_t = \int_0^t [\alpha_s, v] ds - t \int_0^1 [\alpha_s, v] ds$ ($v \in \mathfrak{g}$), $b(t, \gamma) = \int_0^t (R_{\gamma_s})_*^{-1} \circ d\gamma_s \in \mathfrak{g}$. Here $[\cdot, \cdot]$ denotes the Lie bracket. Also $(\square \alpha, h)$ denotes the coupling of $\square \alpha(\gamma) \in H_0^*$ and $h \in H_0$.

For simplicity we denote

$$\square = \nabla_{\mu_e}^* \nabla + I + T_{b(1)} + T_2 + T_3,$$

where T_2, T_3 are 0-order operators acting on 1-forms corresponding to the terms $\int_0^1 ((C\alpha)_t, h_t) dt$ and $-\int_0^1 \int_0^1 (C\alpha_t, h_s) dt ds$ respectively.

Proof of Theorem 2.4. Let $\alpha \in L^2(\wedge^1 T^* L_e(G))$ and assume that $\square \alpha = 0$. We need to show that $\alpha \in \cap_{1 < p < 2} \mathbb{D}^{\infty, p}(\wedge^1 T^* L_e(G))$. Let $\theta \in \mathfrak{F} C_b^\infty(\wedge^1 T^* L_e(G))$. Then

$$\begin{aligned} (\alpha, \nabla_{\nu_e}^* \nabla \theta) &= (\alpha, (\square - I - T_{b(1)} - T_2 - T_3) \theta) \\ &= - \left((I + T_{b(1)}^* + T_2^* + T_3^*) \alpha, \theta \right). \end{aligned} \quad (8.14)$$

Since $b(1) \in \cap_{p > 1} L^p(L_e(G), d\nu_e)$, the weak derivative $\nabla_{\nu_e}^* \nabla \alpha$ belongs to $\cap_{1 < p < 2} L^p(\wedge^1 T^* L_e(G))$. Hence by Theorem 2.16 in [2], $\alpha \in \cap_{1 < p < 2} \mathbb{D}^{2, p}(\wedge^1 T^* L_e(G))$ which implies $\alpha \in \cap_{1 < p < 2} \mathbb{D}^{\infty, p}(\wedge^1 L_e(G))$. Also note that $d\alpha = 0$. Let f and f^K be the function in Theorem 2.2 Then $df^K = \psi'_K(f) \alpha$ on $L_e(G)$. Note that α satisfies the equation $d^* \alpha = 0$ on $L_e(G)$. Hence we have

$$\begin{aligned} \int_{L_e(G)} |\alpha(\gamma)|_{T_\gamma L_e(G)}^2 d\nu_e(\gamma) &= \lim_{K \rightarrow \infty} \int_{L_e(G)} (\alpha(\gamma), \psi'_K(f) \alpha(\gamma))_{T_\gamma L_e(G)} d\nu_e(\gamma) \\ &= \lim_{K \rightarrow \infty} \int_{L_e(G)} d^* \alpha(\gamma) f^K(\gamma) d\nu_e(\gamma) \\ &= 0. \end{aligned}$$

This implies $\alpha = 0$ which proves $\ker \square = \{0\}$. We prove (2.5). Let $H_1 = \overline{\{df \mid f \in \mathfrak{F} C_b^\infty(L_e(G))\}}$ and $H_2 = \overline{\{d^* \alpha \mid \alpha \in \mathfrak{F} C_b^\infty(\wedge^2 T^* L_e(G))\}}$. It is easy to see $H_1 \cap H_2 = \{0\}$. Let $H_3 = (H_1 \oplus H_2)^\perp$. Assume there exists a non-zero $\alpha \in H_3$. Then for any smooth cylindrical 1-form β ,

$$(\square \beta, \alpha)_{L^2(\wedge^1 T^* L_e(G))} = (dd^* \beta, \alpha) + (d^* d \beta, \alpha).$$

Since $d^* \beta$ and $d\beta$ can be approximated by smooth cylindrical functions and 1-forms respectively, we obtain $(\square \beta, \alpha) = 0$. This shows $\square \alpha = 0$ in weak sense. By the essential-selfadjointness of $(\square, \mathfrak{F} C_b^\infty(\wedge^1 T^* L_e(G)))$ which is due to [35], this implies $\alpha \in D(\square)$ and $\square \alpha = 0$. Hence $\alpha = 0$ which completes the proof. \square

We give a proof of Weitzenböck formula for the sake of completeness. The reader may find the proof in [11]. Also we note that this calculation is essentially similar to that of Γ_2 of the Dirichlet form in [17, 34]. First we recall some results in [2].

Lemma 8.4. *Let X_h be the right-invariant vector field corresponding to $h \in H$.*

(1) *We have*

$$\int_{L_e(G)} X_h f \cdot g d\nu_e = \int_{L_e(G)} f \cdot (-X_h g + (h, b)g) d\nu_e.$$

Here $(h, b) = \int_0^1 (\dot{h}(s), db(s))$.

(2) *For any $h, k \in H$,*

$$\nabla_{X_h} X_k = X_{-P_0 \int_0^1 [h_s, \dot{k}_s] ds},$$

where $P_0 h = h_t - th_1$.

(3) *For any $h, k \in H_0$,*

$$[X_h, X_k] = X_{[k, h]},$$

where $[X_h, X_k]$ is the Lie bracket of the vector field on $L_e(G)$.

Proof of Lemma 8.3. We fix a complete orthonormal system $\{e_i\}$ of H_0 . By Lemma 8.4, for any smooth 1-form α on $L_e(G)$,

$$d^* \alpha = \sum_i (-X_{e_i}(\alpha(e_i)) + (e_i, b)\alpha(e_i)),$$

where $\alpha(e_i)$ stands for the coupling of $\alpha(\gamma) \in H_0^*$ and $e_i \in H_0$. Let β be a smooth 2-form on $L_e(G)$. By Lemma 8.4,

$$(d^* \beta)(e_k) = - \sum_i X_{e_i}(\beta(e_i, e_k)) + \sum_i (e_i, b)\beta(e_i, e_k) - \sum_{i < j} \beta(e_i, e_j) ([e_j, e_i], e_k).$$

Using these, we have for $h \in H_0$

$$\begin{aligned} ((d^* d + d d^*) \alpha)(h) &= - \sum_i X_{e_i}(X_{e_i}(\alpha(h))) + \sum_i (e_i, b)X_{e_i}(\alpha(h)) + \alpha(h) \\ &\quad + \sum_{i < j} \alpha([e_j, e_i])([e_j, e_i], h) + \alpha\left(P_0 \int_0^1 [h_s, db_s]\right) - \sum_i (e_i, b)\alpha([e_i, h]) \\ &\quad + \sum_i X_{[h, e_i]}(\alpha(e_i)) + \sum_i X_{e_i}(\alpha([h, e_i])) - \sum_{i < j} (X_{e_i}(\alpha(e_j)) - X_{e_j}(\alpha(e_i)))([e_j, e_i], h) \end{aligned}$$

By the definition of the covariant derivative, we have

$$\begin{aligned} (\nabla_{\nu_e}^* \nabla \alpha)(h) &= - \sum_i X_{e_i}(X_{e_i}(\alpha(h))) + \sum_i (e_i, b)X_{e_i}(\alpha(h)) - \sum_i (e_i, b)\alpha(\nabla_{e_i} h) \\ &\quad + 2 \sum_i X_{e_i}(\alpha(\nabla_{e_i} h)) - \sum_i \alpha(\nabla_{e_i} \nabla_{e_i} h), \end{aligned}$$

where $\nabla_h k = -P_0 \left(\int_0^\cdot [h_s, \dot{k}_s] ds \right)$ for $h, k \in H_0$. Hence

$$\begin{aligned} ((d^* d + d d^*) \alpha)(h) &= (\nabla_{\nu_e}^* \nabla \alpha)(h) + \alpha(h) + \sum_i \alpha(\nabla_{e_i} \nabla_{e_i} h) \\ &\quad + \frac{1}{2} \sum_{i,j} \alpha([e_j, e_i]) ([e_j, e_i], h) + I_1 + I_2. \end{aligned}$$

Here

$$\begin{aligned} I_1 &= \alpha \left(P_0 \int_0^\cdot [h_s, db_s] \right) - \sum_i (e_i, b) \alpha([e_i, h]) + \sum_i (e_i, b) \alpha(\nabla_{e_i} h), \\ I_2 &= \sum_i X_{[h, e_i]} (\alpha(e_i)) + \sum_i X_{e_i} (\alpha([h, e_i])) - \sum_{i < j} (X_{e_i} (\alpha(e_j)) - X_{e_j} (\alpha(e_i))) ([e_j, e_i], h) \\ &\quad - 2 \sum_i X_{e_i} (\alpha(\nabla_{e_i} h)). \end{aligned}$$

By the explicit calculation, $I_1 = (T_{b(1)} \alpha)(h)$ and $I_2 = 0$. We calculate $\frac{1}{2} \sum_{i,j} \alpha([e_j, e_i]) ([e_j, e_i], h)$ and $\sum_i \alpha(\nabla_{e_i} \nabla_{e_i} h)$.

$$\begin{aligned} \sum_i \alpha(\nabla_{e_i} \nabla_{e_i} h) &= \sum_i \int_0^1 \left(\dot{\alpha}_t, - \left[e_i(t), [e_i(t), \dot{h}_t] - \int_0^1 [e_i(s), \dot{h}_s] ds \right] \right) dt \\ &= - \sum_i \int_0^1 \left([\dot{\alpha}_t, e_i(t)], [\dot{h}_t, e_i(t)] \right) dt \\ &\quad + \sum_i \left(\int_0^1 [\dot{\alpha}_t, e_i(t)] dt, \int_0^1 [\dot{h}_s, e_i(s)] ds \right). \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \sum_{i,j} \alpha([e_j, e_i]) ([e_j, e_i], h) &= \sum_i \int_0^1 \left([\dot{\alpha}_t, e_i(t)] - \int_0^1 [\dot{\alpha}_t, e_i(t)] dt, [\dot{h}_t, e_i(t)] - \int_0^1 [\dot{h}_t, e_i(t)] dt \right) dt \\ &\quad - \sum_i \int_0^1 \left([\dot{\alpha}_t, e_i(t)] - \int_0^1 [\dot{\alpha}_t, e_i(t)] dt, \int_0^t [\dot{h}_s, \dot{e}_i(s)] ds - \int_0^1 \left(\int_0^u [\dot{h}_s, \dot{e}_i(s)] ds \right) du \right) dt. \end{aligned}$$

Thus

$$\begin{aligned} \sum_i \alpha(\nabla_{e_i} \nabla_{e_i} h) + \frac{1}{2} \sum_{i,j} \alpha([e_j, e_i]) ([e_j, e_i], h) &= - \sum_i \int_0^1 \left([\dot{\alpha}_t, e_i(t)] - \int_0^1 [\dot{\alpha}_t, e_i(t)] dt, \int_0^t [\dot{h}_s, \dot{e}_i(s)] ds \right) dt \\ &= - \sum_{i=1}^d \left(\int_0^1 [[\alpha_t, \varepsilon_i], \varepsilon_i] dt, \int_0^1 h_t dt \right) - \sum_{i=1}^d \int_0^1 ([\alpha_t, \varepsilon_i], [h_t, \varepsilon_i]) dt \\ &= - \left(\int_0^1 (C\alpha)_t dt, \int_0^1 h_t dt \right) + \int_0^1 (C\alpha_t, h_t) dt. \end{aligned}$$

This completes the proof. \square

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